QUESTION

(i) Show that the function

$$f(x) = x^2 e^{-x}$$

has two stationary points and determine their nature.

(ii) Determine the points of inflection of the function $f(x)$.

(iii) Given that $x = 1$ is an approximate solution of the equation $3x^2e^{-x} = 1$ use the Newton Raphson formula ONCE to obtain a better approximation (correct to four decimal place).

ANSWER

(i)

$$f(x) = x^2 e^{-x}$$
$$\frac{df}{dx} = x^2(-e^{-x}) + e^{-x}(2x) = (2x - x^2)e^{-x}$$

There is a stationary point when $\frac{df}{dx} = 0$, i.e. $2x - x^2 = 0$ $(e^{-x} \neq 0)$ therefore $x(2-x) = 0$, therefore $x = 0$ or $x = 2$

$$\begin{aligned}\frac{d^2f}{dx^2} &= (2x - x^2)(-e^{-x}) + e^{-x}(2 - 2x)\\ &= e^{-x}(-2x + x^2 + 2 - 2x)\\ &= (x^2 - 4x + 2)e^{-x}\end{aligned}$$

When $x = 0$, $\frac{d^2f}{dx^2} = 2e^{-0} = 2 > 0$ therefore f has a minimum at $x = 0$

When $x = 2$, $\frac{d^2f}{dx^2} = (2^2 - 4(2) + 2)e^{-2} = -\frac{2}{e^2} < 0$ therefore f has a maximum at $x = 2$

(ii) $\frac{d^2f}{dx^2} = 0$ when $x^2 - 4x + 2 = 0$, $x = \frac{4 \pm \sqrt{((-4)^2 - 4(1)2)}}{2} = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$

$$\begin{aligned}\frac{d^3f}{dx^3} &= (x^2 - 4x + 2)(-e^{-x} + (2x - 4)e^{-x}\\ &= e^{-x}(-x^2 + 4x - 2 + 2x - 4)\\ &= e^{-x}(-x^2 + 6x - 6)\\ &= -(x^2 - 6x + 6)e^{-x}\end{aligned}$$

When $x = 2+\sqrt{2}$, $\dfrac{d^3f}{dx^3} \sim 2.83e^{-(2+\sqrt{2})} \neq 0$

When $x = 2-\sqrt{2}$, $\dfrac{d^3f}{dx^3} \sim -2.83e^{-(2-\sqrt{2})} \neq 0$

Therefore there are inflection points at $x = 2+\sqrt{2}$ and $x = 2-\sqrt{2}$

(iii) $3x^2e^{-x} = 1,\ g(x) = 3x^2e^{-x} - 1 = 0,\ \dfrac{dg}{dx} = 3\dfrac{df}{dx} = 3(2x - x^2)e^{-x}$

$x_1 = 1 - \dfrac{g(1)}{g'(1)} = 1 - \dfrac{0.1036}{2.2073} = 0.9531$