QUESTION

(a) Write $\cos(2x) - \cos(4x)$ as a product of trigonometric functions, and hence deduce ALL solutions of the equation

$$\cos(2x) = \cos(4x).$$

(b) Differentiate with respect to x the following functions

(i) $\dfrac{x}{\sqrt{1+x^2}}$, (ii) $\exp(x^2 \sinh x)$, (iii) $\tan^3\{(1-x^2)^2\}$.

ANSWER

(a)

$$\begin{aligned}
\cos(2x) - \cos(4x) &= -2\sin\left(\frac{2x+4x}{2}\right)\sin\left(\frac{2x-4x}{2}\right) \\
&= -2\sin(3x)\sin(-x) \\
&= 2\sin(3x)\sin(x) \\
\cos(2x) = \cos(4x) &\Rightarrow \cos(2x) - \cos(4x) = 0 \\
&\text{i.e. } 2\sin(3x)\sin(x) = 0 \\
\text{therefore } \sin x = 0 &\text{ or } \sin 3x = 0
\end{aligned}$$

$\sin x = 0 \Rightarrow x = n\pi$; i.e. $x = 0, \pm\pi, \pm 2\pi \ldots$
$\sin(3x) = 0, \Rightarrow 3x = n\pi$, therefore $x = \dfrac{n\pi}{3}$ where n is any integer.
Both conditions are satisfied by $x = \dfrac{n\pi}{3}$ where n is any integer.

(b) (i)

$$\begin{aligned}
\frac{d}{dx}\left\{\frac{x}{\sqrt{1+x^2}}\right\} &= \frac{\sqrt{(1+x^2)}.1 - x\left\{\frac{1}{2}(1+x^2)^{-\frac{1}{2}}(2x)\right\}}{\left\{\sqrt{(1+x^2)}\right\}^2} \\
&= \frac{\sqrt{(1+x^2)} - \frac{x^2}{\sqrt{(1+x^2)}}}{1+x^2} \\
&= \frac{\frac{(1+x^2)-x^2}{\sqrt{(1+x^2)}}}{1+x^2} \\
&= \frac{1}{(1+x^2)^{\frac{3}{2}}}
\end{aligned}$$

(ii)

$$\begin{aligned}
\frac{d}{dx}\left\{e^{x^2\sinh x}\right\} &= e^{x^2\sinh x}\left\{x^2\cosh x + 2x\sinh x\right\} \\
&= x(x\cosh x + 2\sinh x)e^{x^2\sinh x}
\end{aligned}$$

(iii)

$$\begin{aligned}
&\frac{d}{dx}\left\{\tan^3((1-x^2))\right\} \\
= \;& 3\tan^2((1-x^2)^2)\sec^2((1-x^2)^2)2(1-x^2)(-2x) \\
= \;& -12x(1-x^2)\tan^2((1-x^2)^2)\sec^2((1-x^2)^2)
\end{aligned}$$