Question

For each of the following functions, the same as in Exercise 8.11, use Rolle's theorem or the mean value theorem to determine whether the solutions described in Exercise 8.11 are the only ones.

1. $f(x) = 0$ on the interval $[-a, a]$, where a is an arbitrary positive real number and $f(x) = x^{1995} + 7654x^{123} + x$;

2. $\tan(x) = e^{-x}$ for x in $[-1, 1]$;

3. $3\sin^2(x) = 2\cos^3(x)$ for $x > 0$;

4. $3 + x^5 - 1001x^2 = 0$ for $x > 0$;

Answer

1. we know that there is one solution to $f(x) = 0$ in $[-a, a]$, namely $x = 0$ (which can be found with using the intermediate value theorem or by inspection). To see that there are no others, we again use Rolle's theorem: if there were b in $[-a, a]$, $b \neq 0$, with $f(b) = 0$, then there would exist some point c between b and 0 with $f'(c) = 0$. However, $f'(x) = 1995x^{1994} + 941442x^{122} + 1$ and so $f'(c) \geq 1 > 0$ for all $c \in \mathbf{R}$. Hence, by Rolle's theorem, there is no second solution to $f(x) = 0$.

2. again working with $g(x) = \tan(x) - e^{-x}$, we saw earlier that there is a solution to $g(x) = 0$ in the interval $[-1, 1]$. However, since $g'(x) = \sec^2(x) + e^{-x} > 0$ for all $x \in (-1, 1)$, Rolle's theorem implies that there can be no second solution to $g(x) = 0$ in the interval $[-1, 1]$. (It is the same reasoning as before: if there were two solutions to $g(x) = 0$, then there would exist a point c between them at which $g'(c) = 0$; however, the calculation above shows that $g'(c) \neq 0$ for all c in $(-1, 1)$)

3. we don't have enough information to decide whether we've found all the solutions to $f(x) = 0$. With $f(x) = 3\sin^2(x) - 2\cos^3(x)$, we have that $f'(x) = 6\sin(x)\cos(x) + 6\cos^2(x)\sin(x) = 6\sin(x)\cos(x)(1 + \cos(x)) = 0$ when $x = k\pi$ for $k \in \mathbf{N}$ (since $\sin(k\pi) = 0$) and when $x = (k + \frac{1}{2})\pi$ (since $\cos((k + \frac{1}{2})\pi) = 0$ for $k \in \mathbf{N}$). Note that $f(k\pi) = -2\cos^3(k\pi) = (-1)^{k+1}2 \neq 0$ and that $f((k + \frac{1}{2})\pi) = 3\sin^2((k + \frac{1}{2})\pi) = 3 \neq 0$. So, for any $m \in \mathbf{N}$, consider the interval $(m\pi, (m + 2)\pi)$.

   So, there exist three points in this interval at which $f'(x) = 0$, namely at $(m + \frac{1}{2})\pi$, $(m + 1)\pi$, and $(m + \frac{3}{2})\pi$, and our earlier analysis using the intermediate value theorem found only two points in this interval at which $f(x) = 0$. However, while Rolle's theorem yields that two points at which $f(x) = 0$ yields one point at which $f'(x) = 0$, we are unable to argue the other way: there may be many points at which $f'(x) = 0$ and still no points at which $f(x) = 0$. This example shows the limitations of this sort of analysis.

4. for $f(x) = 3 + x^5 - 1001x^2$ on $x > 0$, again differentiate: $f'(x) = 5x^4 - 2002x = x(5x^3 - 2002)$, and so there is only one point in $(0, \infty)$ at which $f'(x) = 0$, namely the solution c of $5c^3 - 2002 = 0$. By calculation, we have that $c = 7.3705...$, and so if there is a second solution to $f(x) = 0$ in $(0, \infty)$, it must lie in the interval $(0, c)$ (since by Rolle's theorem, if there are two solutions to $f(x) = 0$, then there exists at least one solution to $f'(x) = 0$ between them).

   Since $f(0) = 3$ and since $f(c) = -32624.3179...$, the intermediate value property implies that that there is a solution to $f(x) = 0$ in the interval $(0, c)$. Since the only solution to $f'(x) = 0$ on $(0, \infty)$ occurs at c, Rolle's theorem implies that there can be at most two solutions to $f(x) = 0$ in $(0, \infty)$, and we have found them both.