Question

Show that the points $z = 1$, $z = -\frac{1}{2}$ of the z-plane are inverse for the circle C_1 with centre -1 and radius 1.

With the circle centre 1 and the radius 1 denoted by C_2 find a Mobius transformation

$$w = \frac{az + b}{cz + d} \qquad (ad \neq bc)$$

which transforms $z = 1$ to $w = -1$, C_1 to $\mathrm{Re}(w) = \frac{1}{2}$ and C_2 to $\mathrm{Re}(w) = -\frac{1}{2}$.

Answer

DIAGRAM

ABC are collinear $AB = \frac{1}{2}$, $AC = 2$ $AO = 1$, therefore $AB.AC = AO^2$

So C and B are inverse with respect to C_1

Bilinear transformations map circles and inverse points to circles and inverses or lines and images.

So since $C_2 \to L_2$ and B' and A are inverse with respect to C_2

$A \to C \Rightarrow B' \to w = -2$

Also since $C_1 \to L_1$ and C and B are inverse with respect to C_1

$C \to A \Rightarrow B \to w = 2$

So we have

z	w
1	-1
$-\frac{1}{2}$	2
$\frac{1}{2}$	2

So since $czw + dw - az - b = 0$, we have

$$-c - d - a - b = 0 \qquad (1)$$

$$-c + 2d + \frac{1}{2}a - b = 0 \qquad (2)$$

$$-c - 2d - \frac{1}{2}a - b = 0 \qquad (3)$$

$$\left.\begin{array}{l} (2) - (1) \to 3d + \frac{3}{2}a = 0 \\ (3) - (1) \to -d + \frac{1}{2}a = 0 \end{array}\right\} \Rightarrow a = d = 0 \qquad \text{so } -c - b = 0$$

i.e. $w = -\dfrac{1}{z}$ - inversion in the origin, and reflections.