Question

Use the calculus of residues to show that

a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ when $a>b>0$, and

ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2-a^2} = \frac{1}{2a^2} - \frac{\pi}{2a}\cot\pi a$ when a is not an integer.

Answer

a) Let $z=e^{i\theta}$ $\qquad d\theta = \dfrac{dz}{iz}$ $\qquad \cos\theta = \frac{1}{2}(z+\frac{1}{z})$

C is the unit circle.

$$I = \int_0^2 \pi \frac{d\theta}{a+b\cos\theta} \qquad a>b>0$$

$$= \int_C \frac{dz}{iz\left(a+\frac{b}{2}\left(z+\frac{1}{z}\right)\right)} = \frac{2}{ib}\int_C \frac{dz}{z^2+\frac{2a}{b}z+1}$$

The integrand has simple poles at

$z = -\dfrac{a}{b} + \sqrt{\dfrac{a^2}{b^2}-1} = \alpha_1$ - inside C

$z = -\dfrac{a}{b} - \sqrt{\dfrac{a^2}{b^2}-1} = \alpha_2$ - outside C

The residue at $z=\alpha_1$ is $\dfrac{1}{\alpha_1-\alpha_2} = \dfrac{b}{2\sqrt{a^2-b^2}}$

So $I = \dfrac{2}{ib}2\pi i\dfrac{b}{2\sqrt{a^2-b^2}} = \dfrac{2\pi}{\sqrt{a^2-b^2}}$

b) Let $f(z) = \dfrac{1}{z^2 - a^2} \qquad a \notin Z$

Apply the usual method with $\pi \cot \pi z f(z)$, which has simple poles at $z = \pm a$.

$\text{res}(a) = \dfrac{\pi \cot \pi a}{2a} = \text{res}(-a)$

So $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{n^2 - a^2} = -\frac{\pi \cot \pi a}{a}$

So $\displaystyle 2\sum_{n=1}^{\infty} \frac{1}{n^2 - a^2} + \frac{1}{-a^2} = -\frac{\pi \cot \pi a}{a}$

Hence $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 - a^2} = \frac{1}{2a^2} - \frac{\pi \cot \pi a}{2a}$