Question
For the following system of equations

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(a) Write down the matrix and the augmented matrix

(b) Find the rank of both by the elimination method

(c) Use this information to determine whether the equations have a solution, and if they do how many free variables there are.

(d) If they do have a solution, find it, and confirm that indeed it has the right number of free variables.

Answer

(a) $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix} \qquad A : b = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix}$

(b) Use elimination method to find rank

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix} \rightarrow \text{(row 2} \rightarrow \text{row 2} - \text{row 1)}$$

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix} \rightarrow \text{(row 3} \rightarrow \text{row 3} + \text{row 2)}$$

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence both A and $A : b$ have rank 2

(c) Hence equations do have a solution and since $r(A) = r(A : b)$

No of free parameters = no of unknowns $- \, r(A) = 3 - 2 = 1$

(d) Equations are

$$\begin{aligned} x + 2y + z &= 0 \\ -y - 2z &= 0 \end{aligned}$$

Let $z = C \Rightarrow y = -2C \Rightarrow x = -2y - z = 4C - C = 3C$ and $\mathbf{x} = \begin{bmatrix} 3C \\ -2C \\ C \end{bmatrix}$ with one free variable as expected.