Question

For the following system of equations

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ 3 \end{pmatrix}$$

(a) Write down the matrix and the augmented matrix

(b) Find the rank of both by the elimination method

(c) Use this information to determine whether the equations have a solution, and if they do how many free variables there are.

(d) If they do have a solution, find it, and confirm that indeed it has the right number of free variables.

Answer

(a) $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \qquad A : b = \begin{pmatrix} 1 & 2 & 1 & 6 \\ 2 & 1 & -1 & 7 \\ 1 & -1 & 0 & 3 \end{pmatrix}$

(b) Use elimination method to find rank

$$\begin{pmatrix} 1 & 2 & 1 & 6 \\ 2 & 1 & -1 & 7 \\ 1 & -1 & 0 & 3 \end{pmatrix} \rightarrow \text{(exchange rows } 1, 2)$$

$$\begin{pmatrix} 2 & 1 & -1 & 7 \\ 1 & 2 & 1 & 6 \\ 1 & -1 & 0 & 3 \end{pmatrix} \rightarrow \begin{matrix} (\text{row } 2 \rightarrow 2\text{row } 2 - \text{row } 1) \\ (\text{row } 3 \rightarrow 2\text{row } 3 - \text{row } 1) \end{matrix}$$

$$\begin{pmatrix} 2 & 1 & -1 & 7 \\ 0 & 3 & 3 & 5 \\ 0 & -3 & 1 & -1 \end{pmatrix} \rightarrow (\text{row } 3 \rightarrow \text{row } 3 + \text{row } 2)$$

$$\begin{pmatrix} 2 & 1 & -1 & 7 \\ 0 & 3 & 3 & 5 \\ 0 & 0 & 4 & 4 \end{pmatrix}$$

Hence both $r(A) = r(A : b) = 3$

(c) Hence equations do have a solution and since $r(A) = r(A : b)$, no. of free parameters = no of unknowns $- \, r(A) = 3 - 3 = 0$

(d) Equations are

$$\begin{aligned} 2x + y - z &= 7 \\ 3y + 3z &= 5 \\ 4z &= 4 \end{aligned}$$

Let $z = 1 \Rightarrow y = \frac{2}{3} \Rightarrow 2x = 7 + z - y = 7 + 1 - \frac{2}{3} = \frac{22}{3} \Rightarrow x = \frac{11}{3}$ and $\mathbf{x} = \begin{pmatrix} \frac{11}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix}$ with no free variable as expected.