Question

For the following system of equations

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 2 & 0 & 0 \\ 0 & 3 & -2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 1 \\ 1 \end{pmatrix}$$

(a) Write down the matrix and the augmented matrix

(b) Find the rank of both by the elimination method

(c) Use this information to determine whether the equations have a solution, and if they do how many free variables there are.

(d) If they do have a solution, find it, and confirm that indeed it has the right number of free variables.

Answer

(a) $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 2 & 0 & 0 \\ 0 & 3 & -2 & -2 \end{pmatrix}$ $\quad A : b = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & -1 & 4 \\ 1 & 2 & 0 & 0 & 1 \\ 0 & 3 & -2 & -2 & 1 \end{pmatrix}$

(b) Use elimination method to find rank

$$A : b = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & -1 & 4 \\ 1 & 2 & 0 & 0 & 1 \\ 0 & 3 & -2 & -2 & 1 \end{pmatrix} \rightarrow \begin{array}{l} (\text{row } 2 \rightarrow \text{row } 2 - \text{row } 1) \\ (\text{row } 3 \rightarrow \text{row } 3 - \text{row } 1) \end{array}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & -2 & 4 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 3 & -2 & -2 & 1 \end{pmatrix} \rightarrow \ (\text{row } 4 \rightarrow \text{row } 4 - 3\text{row } 3)$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & -2 & 4 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -2 \end{pmatrix} \rightarrow \begin{array}{l} (\text{row } 3 \rightarrow \text{row } 2) \\ (\text{row } 2 \rightarrow \text{row } 4) \\ (\text{row } 4 \rightarrow \text{row } 3) \end{array}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & -2 & -2 & 4 \end{pmatrix} \rightarrow \ (\text{row } 4 \rightarrow \text{row } 4 + 2\text{row } 3)$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence both $r(A) = r(A : b) = 3$

(c) Hence equations do have a solution and since $r(A) = r(A : b)$, no. of free parameters = no of unknowns $- r(A) = 4 - 3 = 1$

(d) Equations are

$$\begin{aligned} x + y + z + w &= 0 \\ y - z - w &= 1 \\ z + w &= -2 \end{aligned}$$

Let $w = C \Rightarrow z = -2 - C \Rightarrow y = -1 \Rightarrow x = 3$ and

$\mathbf{x} = \begin{pmatrix} 3 \\ -1 \\ -2 - C \\ C \end{pmatrix}$ with one free variable as expected.