Question

Find the eigenvectors for the following matrices

(a) $\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

Answer

(a) $\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$

To find eigenvalues

$$\begin{aligned} \begin{vmatrix} 3-\lambda & 4 \\ 2 & 1-\lambda \end{vmatrix} &= (\lambda-3)(\lambda-1)-8 \\ &= \lambda^2 - 4\lambda + 3 - 8 \\ &= \lambda^2 - 4\lambda + 5 \\ &= (\lambda-5)(\lambda+1) = 0 \end{aligned}$$

$\Rightarrow$ eigenvalues are $\lambda_1 = 5, \lambda_2 = -1$

eigenvector let $\mathbf{e_1} = \begin{pmatrix} e_{11} \\ e_{12} \end{pmatrix}$

and $A\mathbf{e_1} = \lambda_1 \mathbf{e_1}$ implies $A\begin{pmatrix} e_{11} \\ e_{12} \end{pmatrix} = \lambda_1 \begin{pmatrix} e_{11} \\ e_{12} \end{pmatrix}$

From the first equation we get

$3e_{11} + 4e_{12} = 5e_{11} \Rightarrow 4e_{12} = 2e_{11} \Rightarrow 2e_{12} = e_{11}$

The second equation will be equivalent.

Let $e_{12} = c$

The first eigenvector is $\mathbf{e_1} = \begin{pmatrix} 2c \\ c \end{pmatrix}$

For the second eigenvector let $\mathbf{e_2} = \begin{pmatrix} e_{21} \\ e_{22} \end{pmatrix}$

and $A\mathbf{e_2} = \lambda_2 \mathbf{e_2} \Rightarrow \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} e_{21} \\ e_{22} \end{pmatrix} = - \begin{pmatrix} e_{21} \\ e_{22} \end{pmatrix}$

From the first equation we get

$3e_{21} + 4e_{22} = -e_{21} \Rightarrow 4e_{22} = -4e_{21} \to -e_{22} = e_{21} = -c$

Hence $\mathbf{e_2} = \begin{pmatrix} -c \\ c \end{pmatrix}$ is the eigenvector corresponding to $\lambda_2 = -1$

(b) $B\mathbf{x} = \lambda \mathbf{x}$

$$B = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

To find eigenvectors $|B - \lambda I| = 0$

$$\begin{aligned} \begin{vmatrix} 1-\lambda & 0 & 1 \\ -1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} &= (1-\lambda)^3 - (1-\lambda) \\ &= (1-\lambda)[(1-\lambda)^2 - 1] \\ &= (1-\lambda)\lambda(2-\lambda) = 0 \end{aligned}$$

Hence the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$

To find the eigenvectors:

$$\mathbf{e_1} = \begin{pmatrix} e_{11} \\ e_{12} \\ e_{13} \end{pmatrix} \qquad B\mathbf{e_1} = \lambda_1 \mathbf{e_1} = 0$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{12} \\ e_{13} \end{pmatrix} = 0$$

$$\Rightarrow e_{11} = -e_{13} = e_{12} = C \Rightarrow \mathbf{e_1} = C \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} C \\ C \\ -C \end{pmatrix}$$

$$\mathbf{e_2} = \begin{pmatrix} e_{21} \\ e_{22} \\ e_{23} \end{pmatrix} \qquad B\mathbf{e_2} = \lambda_2 \mathbf{e_2} = \mathbf{e_2}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} e_{21} \\ e_{22} \\ e_{23} \end{pmatrix} = \begin{pmatrix} e_{21} \\ e_{22} \\ e_{23} \end{pmatrix}$$

$$\Rightarrow e_{21}+e_{23}=e_{21}, e_{21}+e_{23}=e_{23}, -e_{21}+e_{22}=e_{22} \Rightarrow e_{21}=e_{23}=0$$

$$\Rightarrow \mathbf{e_2} = \begin{pmatrix} 0 \\ D \\ 0 \end{pmatrix}$$

$$\mathbf{e_3} = \begin{pmatrix} e_{31} \\ e_{32} \\ e_{33} \end{pmatrix} \qquad B\mathbf{e_3} = \lambda_3 \mathbf{e_3} = 2\mathbf{e_3}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} e_{31} \\ e_{32} \\ e_{33} \end{pmatrix} = 2 \begin{pmatrix} e_{31} \\ e_{32} \\ e_{33} \end{pmatrix}$$

$$\Rightarrow e_{31} = -e_{32} = e_{33} = E \Rightarrow \mathbf{e_3} = E \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} E \\ -E \\ E \end{pmatrix}$$