QUESTION
Without using your calculator, find

(i) the least positive residue of 24.17 mod 29.

(ii) the least absolute residue of 19.14 mod 23.

(iii) the remainder when 5^10 is divided by 19.

(iv) the final digit of $1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! + 9! + 10!$

ANSWER
We use least absolute residues, to keep the numbers in the calculations small:

(i) $24.17 \equiv (-5).(-12) \equiv 60 \equiv 2 \bmod 29$.

(ii) $19, 14 \equiv (-4)(-9) \equiv 36 \equiv 13 \bmod 23$. Thus the least positive residue is 13. The least absolute residue is -10.

(iii) $5^2 \equiv 25 \equiv 6 \bmod 19$. Hence $5^4 equiv 36 \equiv -2 \bmod 9$ and $5^10 \equiv (5^4)^2, 5^2 \equiv (-2)^2.6 \equiv 24 \equiv 5 \bmod 19$.

Thus the remainder when 5^10 is divided by 19 is 5.

(iv) The final digit of a number is given by its congruence class mod 10. (e.g. $1527 = 1.10^3 + 5.10^2 + 2.10 + 7$ so is congruent to 7 mod 10.)

Now $n = 1.2.3. \ldots .n$, so if $n \geq 5$, $n!$ is divisible by both 5 and 2, and hence by 10. Hence $n! \equiv 0 \bmod 10$ for all $n \geq 5$.

Thus $1! + 2! + \ldots + 10! \equiv 1! + 2! + 3! + 4! \equiv 1 + 2 + 6 + 24 \equiv 33 \equiv 3 \bmod 10$, and so the final digit of $1! + 2! + \ldots + 10!$ is 3.