Question

Suppose that an apartment room contains $40m^3$ of air and that it is initially free of carbon monoxide. At midnight a smoker enters the room and lights a cigarette that produces carbon monoxide at a rate of $1.2 \times 10^{-6} m^3/sec$. A window is open so that fresh air enters the room at a rate of $3 \times 10^{-3} m^3/sec$. In the room the air and pollutants are quickly mixed by a fan and mixed air then exits through another window at the same rate as it entered. Determine the concentration of carbon monoxide in the room as a function of time. Extended exposure to carbon monoxide at concentrations as low as 0.00012 is harmful to the human body. At what time is this dangerous concentration reached? If a second smoker enters the room 15 minutes after the first smoker how long does it take to reach the danger level?

Answer

Equation for volume of air in room $V(t)$ is:
rate of change of air in room = air flow in - air flow out
$dV/dt = 0$
Hence volume of air in room is constant = 40
Equation for the Carbon dioxide in the room (where M(t) = m^3 of carbon dioxide in the room) is:
Rate of change of M(t) = rate in - rate out
$\frac{dM}{dt} = 1.2^{-6} - 3 \times 10^{-3} \frac{M}{40}$
where $\frac{M}{40}$ is the concentration of carbon dioxide in the room.
Initial condition is $M(0) = 0$
The equation is linear so use integrating factor to give

$$\frac{d}{dt}\left(e^{7.5\times10^{-5}t} M\right) = 1.2 \times 10^{-6} e^{7.5\times10^{-5}t}$$

Using the initial conditions then gives $M = 1.6 \times 10^{-2}\left(1 - e^{-7.5\times10^{-5}t}\right)$
The concentration in the room becomes 0.00012 when $0.00012 = \frac{M}{40} = 4.0^{-4}\left(1 - e^{-7.5\times10^{-5}t}\right)$
which is approximately 1.3hrs.

If the second smoker enters the room after 15minutes then
at t=15mins $M = 1.6 \times 10^{-2} \left(1 - e^{-7.5\times10^{-5}(15\times60)}\right)$
(recall all measurements are in seconds) After this time the equation for the carbon dioxide must include the second smoker so that: $\dfrac{dM}{dt} = 2(1.2^{-6}) - 3 \times 10^{-3}\dfrac{M}{40}$
which has a solution $M = 3.2 \times 10^{-2} \left(1 - Be^{-7.5\times10^{-5}t}\right)$
Again consider when $\dfrac{M}{40} = 0.00012$ to give $t \approx 50$minutes.