Question

Let $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 2\mathbf{j} - \mathbf{k}$. Find the relation that must hold between x_1, x_2 and x_3 if the vector $\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ is to be written as

$$\mathbf{x} = s\mathbf{a} + t\mathbf{b}$$

where s and t are scalars. Show that the vector $\mathbf{c} = 3\mathbf{i}+\mathbf{j}+4\mathbf{k}$ can be written as $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$ and find s and t in this case.

Answer

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 0 & 2 & -1 \end{vmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}, \mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b}) = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$$

Note that $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0 = \mathbf{b} \cdot (\mathbf{a} \times \mathbf{b})$

Take the dot product of the equation $\mathbf{x} = s\mathbf{a} + t\mathbf{b}$ with the vector $\mathbf{a} \times \mathbf{b}$

$\mathbf{x} \cdot (\mathbf{a} \times \mathbf{b}) = s\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) + t\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$

Hence the relation is: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = 0$

or $-3x_1 + x_2 + 2x_3 = 0$

Note that the points which can be written as $\mathbf{x} = s\mathbf{a}+t\mathbf{b}$ all lie on a plane through the origin with equation $-3x + y + 2z = 0$. The vector $\mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ can be written in the form $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$ because the components of $\mathbf{c}$ satisfy the relation: $-3(3) + (1) + 2(4) = -9 + 1 + 8 = 0$

$$\mathbf{c} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 4 \\ 1 & 1 & 1 \end{vmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}, \mathbf{c} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 4 \\ 0 & 2 & -1 \end{vmatrix} = \begin{pmatrix} -9 \\ 3 \\ 6 \end{pmatrix}$$

To find s take the cross product of $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$ with $\mathbf{b}$: $\mathbf{c} \times \mathbf{b} = s(\mathbf{a} \times \mathbf{b}) + t(\mathbf{b} \times \mathbf{b}) = s(\mathbf{a} \times \mathbf{b})$ since $\mathbf{b} \times \mathbf{b} = 0$

$\begin{pmatrix} -9 \\ 3 \\ 6 \end{pmatrix} = s \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$ and so $s = 3$

To find t take the cross product of $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$ with $\mathbf{a}$: $\mathbf{c} \times \mathbf{a} = s(\mathbf{a} \times \mathbf{a}) + t(\mathbf{b} \times \mathbf{a}) = t(\mathbf{b} \times \mathbf{a})$ since $\mathbf{a} \times \mathbf{a} = 0$

$\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = t \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$ and so $t = -1$

hence $\mathbf{c} = 3\mathbf{a} - \mathbf{b}$.