Question

Using the cross product, find a unit vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ where:

(i) $\mathbf{u} = \mathbf{i} + \mathbf{j}$ and $\mathbf{v} = \mathbf{j} + 2\mathbf{k}$;

(ii) $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} + \mathbf{j} + 3\mathbf{k}$;

(iii) $\mathbf{u} = \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 5\mathbf{i} + 8\mathbf{j} + 4\mathbf{k}$

Answer

The cross product, $\mathbf{u} \times \mathbf{v}$ gives a vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$:

(a) Let $\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$. The length of $\mathbf{n}$ is $|\mathbf{n}| = \sqrt{2^2 + (-2)^2 + 1^2} = 3$

A unit vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ is

$$\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{-2}{3} \\ \frac{1}{3} \end{pmatrix}.$$

(b) Let $\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ 4 & 1 & 3 \end{vmatrix} = \begin{pmatrix} 5 \\ -5 \\ -5 \end{pmatrix}$. The length of $\mathbf{n}$ is $|\mathbf{n}| = 5\sqrt{3}$

A unit vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ is

$$\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{1}{5\sqrt{3}}\begin{pmatrix} 5 \\ -5 \\ -5 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix}.$$

(c) Let $\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 3 \\ 5 & 8 & 4 \end{vmatrix} = \begin{pmatrix} -20 \\ 15 \\ -5 \end{pmatrix}$. The length of $\mathbf{n}$ is $|\mathbf{n}| = \sqrt{650} = 5\sqrt{26}$

A unit vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ is

$$\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{1}{5\sqrt{26}}\begin{pmatrix} -20 \\ 15 \\ -5 \end{pmatrix} = \begin{pmatrix} \frac{-4}{\sqrt{26}} \\ \frac{3}{\sqrt{26}} \\ \frac{-1}{\sqrt{26}} \end{pmatrix}.$$