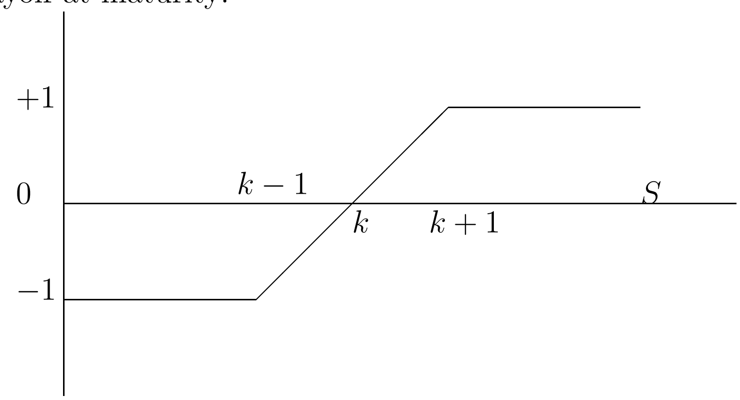
QUESTION

(fiddly!) Using the fundamental solution representation, calculate the continuous-time pricing formula for an option which has a payoff at maturity of

$$payoff(S_r) = \begin{cases} -1, & 0 < S_r < K-1 \\ S_r - K, & K-1 \leq S_r < K+1 \\ +1, & K+1 \leq S_r \end{cases}$$

ANSWER

Payoff at maturity:



Feed into solution of Black-Scholes:

$$V(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_0^{\infty} e^{\frac{\left[\log(\frac{S}{S_1})+(r-\frac{\sigma^2}{2})(T-t)\right]^2}{2\sigma^2(T-t)}} \frac{\text{Payoff}(S')}{S'} ds'$$

Transform to variable $x' = \log S' \Rightarrow dx' = \frac{S'}{S'}$ and payoff becomes:

$$\text{Payoff}e^{x'}) = \begin{cases} -1, & -\infty < x' < \log(k-1) \\ e^{x'} - k, & \log(k-1) < x' < \log(k+1) \\ +1, & \log(k+1) < x' \end{cases}$$

and $-\infty < x' < \infty$

$$V(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \left\{ \int_{\log(K+1)}^{\infty} e^{-\frac{\left[-x'+\log S+(r+\frac{\sigma^2}{2})(T-t)\right]^2}{2\sigma^2(T-t)}} dx' \right. \quad (1)$$

$$+ \int_{\log(k-1)}^{\log(k+1)} e^{-\frac{\left[-x'+\log S+(r-\frac{\sigma^2}{2}(T-t)\right]^2}{2\sigma^2(T-t)}} (e^x - k)\, dx' \quad (2)$$

$$\left. - \int_{-\infty}^{\log(k-1)} e^{-\frac{\left[-x'+\log S+(r-\frac{\sigma^2}{2})(T-t)\right]^2}{2\sigma^2(T-t)}} dx' \right\} \quad (3)$$

Set $X = \log S + \left(r - \frac{\sigma^2}{2}\right)(T-t)$ as shorthand. Isolate and simplify (1), (2) and (3) individually.

$(1) = \int_{\log(k+1)}^{\infty} dx' e^{\frac{(-x'+X)^2}{2\sigma^2(T-t)}}$

Set $y' = \frac{+x'-X}{\sigma\sqrt{(T-t)}}$

$= \int_{\frac{\log(k+1)-X}{\sigma\sqrt{(T-t)}}}^{\infty} dy'\sigma\sqrt{(T-t)}e^{-\frac{y'^2}{2}} = \sigma\sqrt{2\pi(T-t)}\left[1 - N(-d_1^+)\right]$

$d_1^+ = \frac{\log\left(\frac{S}{k+1}\right)+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$

$(3) = -\int_{-\infty}^{\log(k-1)} e^{-\frac{(-x'+X)^2}{2\sigma^2(T-t)}}$

Set$y' = \frac{x'-X}{\sigma\sqrt{T-t}}$

$= -\int_{-\infty}^{\frac{\log(k-1)-X}{\sigma\sqrt{T-t}}} e^{\frac{y'^2}{2}} dy'\sigma\sqrt{T-t} = -\sigma\sqrt{2\pi(T-t)}\left[1 - N(+d_1^-)\right]$

$d_1^- = \log\left(\frac{S}{k-1}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)$

$$\begin{aligned}
(2) &= \int_{\log(k-1)}^{\log(k+1)} dx' e^{-\frac{(-x'+x)^2}{2\sigma^2(T-t)}}\left(e^{x'} - k\right) \\
&= \int_{\log(k-1)}^{\log(k+1)} dx' e^{-\frac{(-x'+X)^2}{2\sigma^2(T-t)}+x'} - k\int_{\log(k-1)}^{\log(k+1)} dx' e^{-\frac{(-x'+X)^2}{2\sigma^2(T-t)}} \\
&= \int_{\log(k-1)}^{\log(k+1)} dx' e^{-\left(\frac{x'^2}{2\sigma^2(T-t)} - 2\frac{\left[X+\sigma^2(T-t)\right]}{2\sigma^2(T-t)}\right)} - k\left[\int_{-d_1^-}^{\infty} + \int_{\infty}^{-d_1^+}\right] e^{-\frac{y'^2}{2}} dy'\sigma\sqrt{T-t} \\
&= e^{-\frac{X^2}{2\sigma^2(T-t)}} \int_{\log(k-1)}^{\log(k+1)} dx' e^{-\frac{\left[x'-(X+\sigma^2(T-t))\right]^2}{2\sigma^2(T-t)} + \frac{\left[X+\sigma^2(T-t)\right]^2}{2\sigma^2(T-t)}} \\
&- k\left[\left[1 - N(-d_1^-)\right] - \left[N(-d_1^+)\right]\right]\sqrt{2\pi(T-t)}\sigma \\
&= e^{X+\frac{\sigma^2(T-t)}{2}} \int_{\frac{\left[\log(k-1)-(X+\sigma^2(T-t))\right]}{\sigma\sqrt{T-t}}}^{\frac{\left[\log(k+1)-(X+\sigma^2(T-t))\right]}{\sigma\sqrt{T-t}}} dy' e^{-\frac{y'^2}{2}}\sigma\sqrt{T-t} \\
&- k\left[N(-d_1^-) - N(-d_1^+)\right]\sqrt{2\pi(t-t)}\sigma
\end{aligned}$$

Now $X + \sigma^2(T-t) = \log S + \left(r + \frac{\sigma^2}{2}\right)(T-t)$

so call $d_2^+ = \log\left(\frac{S}{k+1}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t) = d_1^+ + \sigma^2(T-t)$

and $d_2 + - = \log\left(\frac{S}{k-1}\right) + \left(r + \frac{\sigma^2}{2}\right)(t-t) = d_2^- + \sigma^2(T-t)$

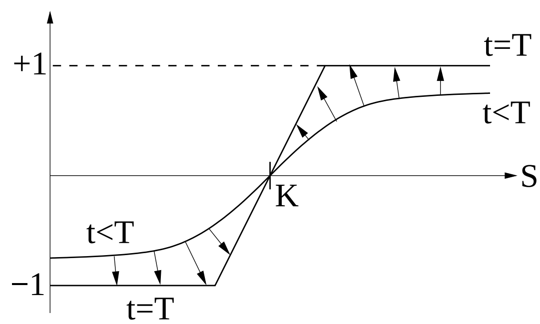
and $X + \frac{\sigma^2(t-t)}{2} = \log S + r(T-t)$

Therefore (2) gives

$$
\begin{aligned}
&= se^{r(T-t)} \int_{-d_2^-}^{-d_2^+} dy' e^{-\frac{y^2}{2}} - k\left[-N(-d_1^-) + N(-d_1^+)\right] \sigma\sqrt{2\pi(T-t)} \\
&= \left\{e^{r(T-t)} S\left[-N(-d_2^+) + N(-d_2^+)\right] - k\left[-N(-d_1^-) + N(-d_1^+)\right]\right\} \times \sigma\sqrt{2\pi(T-t)}
\end{aligned}
$$

But $N(-x) = 1 - N(x)$ (think of probabilities).
Therefore collecting together (1), (2) and (3)

$$
\begin{aligned}
V(S,t) &= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \left\{\sigma\sqrt{2\pi(T-t)} N(d_1^+) - \sigma\sqrt{2\pi(T-t)}(1 - N(d_1^-))\right. \\
&+ \left.\left[se^{r(T-t)}\left[N(d_2^-) - N(d_2^+)\right] - k\left[N(d_1^-) - N(d_1^+)\right]\right] \sigma\sqrt{2\pi(T-t)}\right\} \\
&= e^{-r(T-t)}\left\{-1 - N(d_1^-)(K-1) + N(d_1^+)(K+1)\right\} \\
&+ S\left\{N(d_2^-) - N(d_2^+)\right\}
\end{aligned}
$$



NB. Could also get from the sum of continuous solutions of a portfolio of options with same payoff at maturity. Can you work out which?