## Question

a) Let $a$ be a fixed non-zero complex number. Show that $a^i$ has infinitely many values, and that they all lie on a straight line through the origin in the complex plane.

b) State Cauchy's integral formula expressing the nth derivative $f^{(n)}(a)$ at a point $a$ in terms of a contour integral. Your statement should include conditions under which the formula holds.

Evaluate the following integrals, where $C$ denoted the unit circle:

i) $\displaystyle\int_C \frac{\cos z dz}{z^3}$

ii) $\displaystyle\int_C \frac{e^z dz}{4z^3 - 12z^2 + 9z - 2}$

## Answer

a) $$\begin{aligned} a^i &= \exp(i \log a) = \exp(i(\ln|a| + i(\text{Arg}a + 2n\pi)) \qquad n \in \mathbf{Z} \\ &= \exp(i \ln|a|) \exp(-\text{Arg}a) \exp(-2n\pi) \end{aligned}$$

The first two factors are constants, and the third gives an infinite sequence of distinct real numbers. Hence we have infinitely many different values, all on the line $\arg z = \ln|a|$.

$$\begin{aligned} a^{\sqrt{2}} &= \exp(\sqrt{2} \log a) = \exp(\sqrt{2}(\ln|a| + i(\text{Arg}a + 2n\pi)) \\ &= \exp(\sqrt{2} \ln|a|) \exp(i\sqrt{2} - \text{Arg}a) \exp(i2\sqrt{2}n\pi) \end{aligned}$$

These all lie on the circle, centre 0, radius $|a|^{\sqrt{2}}$. Since the members of the sequence $2\sqrt{2}n\pi$ are all different mod$2\pi$, since $\sqrt{2}$ is irrational, we have infinitely many different values.

b) If $f(z)$ is differentiable inside and on a closed contour $C$, and if $a$ is inside $C$, then

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz$$

i) with $f(z) = \cos z, \;\; a = 0, \;\; n = 2,$

$$\int_C \frac{\cos z}{z^3} dz = \frac{2\pi i}{2!} f''(0) = \pi i(-\cos 0) = -\pi i$$

ii) The denominator factorises as $(z-2)(2z-1)^2 = 4(z-2)(z-\frac{1}{2})^2$

so with $f(z) = \dfrac{e^z}{4(z-2)}, \;\; a = \dfrac{1}{2}, \;\; n = 1$

$$\int_C \frac{e^z dz}{4(z-2)(z-\frac{1}{2})^2} = \frac{2\pi i}{1!} f'\left(\frac{1}{2}\right)$$

Now $f'(z) = \dfrac{(z-2)e^z - e^z}{4(z-2)^2} = \dfrac{(z-3)e^z}{4(z-2)^2}$

So $f'\left(\dfrac{1}{2}\right) = \dfrac{-\frac{5}{2}e^{\frac{1}{2}}}{4\left(-\frac{3}{2}\right)^2} = -\dfrac{5}{18}e^{\frac{1}{2}}.$ So $\displaystyle\int_C = -\frac{5}{9}\pi i e^{\frac{1}{2}}$