Question

a) Show that the function $f(z) = \bar{z}^2 z$ is differentiable only at $z = 0$.

b) Prove that $|\sin z|^2 + |\cos z|^2 \geq 1$, with equality if and only if z is a real number.

c) Evaluate the integral $\int_C \tan z dz$, where C is the straight line segment from $z = 0$ to $z = \frac{1}{2}\pi(1 + i)$. Express your answer in the form $a + ib$ where a and b are real.

Answer

a) $f(z) = \bar{z}^2 z = (x - iy)^2(x + iy) = x^3 + xy^2 + i(-x^2 y - y^3) = u + iv$

Now $\dfrac{\partial u}{\partial x} = 3x^2 + y^2 \qquad \dfrac{\partial v}{\partial y} = -x^2 - 3y^2$

$\dfrac{\partial u}{\partial y} = 2xy \qquad \dfrac{\partial v}{\partial x} = -2xy$

Now $\dfrac{\partial u}{\partial y} \equiv -\dfrac{\partial v}{\partial x}$ but $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ iff $3x^2 + y^2 = -x^2 - 3y^2$ iff $4x^2 = -4y^2$, iff $x = y = 0$.

So the Cauchy-Riemann eqations are satisfied only at $z = 0$. The partial derivatives are continuous there (in fact everywhere), so $f(z)$ is differentiable at $z = 0$.

b) Using the various facts about trigonometric functions we have

$$|\sin z|^2 + |\cos z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y + \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$$

$= \cosh^2 y + \sinh^2 y = \cosh 2y \geq 1$ with equality iff $y = 0$ i.e. if z is real.

c) $\displaystyle\int_C \tan z dz = [-\mathrm{Log}(\cos z)]_0^{(1+i)\frac{\pi}{2}}$ (integral of a continuous derivative)

$= -\mathrm{Log}(\cos(\frac{\pi}{2} + i\frac{\pi}{2})) = -\mathrm{Log}(-\sin(i\frac{\pi}{2}))$

$= -\mathrm{Log}(-i \sinh\frac{\pi}{2}) = -\ln(\sinh\frac{\pi}{2} + i\frac{\pi}{2}$

(Note: $\cos z = cosx \cosh y - i \sin x \sinh y$, so $Re(\cos z) \geq 0$ along C. i.e. $-\frac{\pi}{2} < \arg z \frac{\pi}{2}$ along C, hence we do not encounter singularities for $\text{Log}(\cos z)$. I do not expect this reasoning to appear in the students' answers.)