Question
Find the eigenvalues and eigenfunctions for the differential equation

$$y'' + \lambda y = 0$$

with the following boudary conditions

(a) $y(0) = 0, \qquad y'(1) = 0$

(b) $y'(0) = 0, \qquad y(1) = 0$

(c) $y'(0) = 0, \qquad y'(1) = 0$

(d) $y'(0) = 0, \qquad y'(1) + y(1) = 0$

Answer

(a)

$$y'' + \lambda y = 0 \qquad (*)$$

There are three different cases to consider; $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$. We consider them each in turn.

(i) $\lambda < 0$. In this case we let $\lambda = -k^2$, (where $k \neq 0$) and equation (*) becomes

$y'' - k^2 y = 0, \quad \Rightarrow \quad y = A\cosh(kx) + B\sinh(kx).$

Using the boundary condition $y(0) = 0, \Rightarrow A = 0$, so

$y = B\sinh(kx).$

Differentiating gives $y' = kB\cosh(kx)$ and hence $y'(1) = 0$ gives

$kB\cosh k = 0 \Rightarrow B = 0$ (since $k \neq 0$).

So there is no non-trivial solution if $\lambda < 0$.

(ii) $\lambda = 0$. In this case the equation becomes $y'' = 0, \Rightarrow y = A + Bx$.

The boundary condition $y(0) = 0 \Rightarrow A = 0$, and the boundary condition $y'(1) = 0, \Rightarrow B = 0$. So there is no non-trivial solution for $\lambda = 0$.

(iii) $\lambda > 0$. In this case we let $\lambda = k^2$ (where $k \neq 0$) and equation (*) becomes

$y'' + k^2 y = 0, \quad \Rightarrow \quad y = A\cos(kx) + B\sin(kx).$

The boundary condition $y(0) = 0$ gives $A = 0$, so that $y = B\sin(kx)$.

Hence $y' = Bk\cos(kx)$ and the boundary condition $y'(1) = 0$ gives $Bk\cos k = 0$.

For non-trivial solutions we require $\cos k = 0$ and hence

$$k = \frac{(2n-1)\pi}{2}, \quad n = 1, 2, 3, \ldots.$$

The corresponding eigenvalues and eigenfunctions are

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4}, \quad y_n = \sin\left[\frac{(2n-1)\pi x}{2}\right], \quad n = 1, 2, 3, \ldots.$$

(b) As in part (a) there are no non-trivial solutions unless $\lambda > 0$. We write $\lambda = k^2$ (with $k \neq 0$) and (*) becomes

$y'' + k^2 y = 0$, with solution $y = A\cos(kx) + B\sin(kx)$, and derivative $y' = -Ak\sin(kx) + Bk\cos(kx)$.

Using the boundary condition $y'(0) = 0$ gives $B = 0$ and hence $y = A\cos(kx)$.

Using the boundary condition $y(1) = 0$ gives $A\cos k = 0$ so for non-trivial solutions we require:

$$\cos k = 0, \quad \Rightarrow \quad k = \frac{(2n-1)\pi}{2}, \quad n = 1, 2, 3, \ldots$$

The corresponding eigenvalues and eigenfunctions are

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4}, \quad y_n = \sin\left[\frac{(2n-1)\pi x}{2}\right], \quad n = 1, 2, 3, \ldots.$$

(c) There are no non-trivial solutions when $\lambda < 0$, however in this case there are non-trivial solution when $\lambda = 0$ or $\lambda > 0$.

When $\lambda = 0$ the equation becomes $y'' = 0$ with solution $y = A + Bx$.

Hence $y' = B$ and the boundary condition $y'(0) = 0$ requires $B = 0$. However with this condition the other boundary condition $y'(1) = 0$ is automatically satisfied and hence $y = A$ satisfies the DE and the boundary conditions.

Hence $\lambda_0 = 0$ is an eigenvalue with $y_0 = 1$ the corresponding eigenfunction.

For $\lambda > 0$ the solution is as usual $y = A\cos(kx) + B\sin(kx)$, with derivative $y' = -Ak\sin(kx) + Bk\cos(kx)$.

The boundary condition $y'(0) = 0$ gives $B = 0$ and hence

$y' = -Ak\sin(kx)$

The other boundary condition $y'(1) = 0$ now gives $-Ak\sin k = 0$ so for non-trivial solutions we require:

$\sin k = 0, \quad \Rightarrow \quad k = n\pi, n = 1, 2, 3, \ldots.$

The corresponding eigenvalues and eigenfunctions are

$\lambda_n = n^2\pi^2, \quad y_n = \cos(n\pi) \quad n = 1, 2, 3, \ldots.$

Note that if we allow $n = 0$ this includes the case of the zero eigenvalue.

(d) As in part (a) there are no non-trivial solutions unless $\lambda > 0$. We write $\lambda = k^2$ (with $k \neq 0$) and (*) becomes

$y'' + k^2 y = 0$, with solution $y = A\cos(kx) + B\sin(kx)$, and derivative $y' = -Ak\sin(kx) + Bk\cos(kx)$.

The boundary condition $y'(0) = 0$ gives $B = 0$ so $y = A\cos(kx)$ and $y' = -Ak\sin(kx)$.

Applying the second boundary condition $y(1) + y'(1) = 0$ gives $\cos k - k\sin k = 0$ which implies $k = \cot k$.

By drawing the graphs of $y = \cot x$ and $y = x$ we see that $k = \cot k$ has an infinite number of positive roots; $k_1, k_2, k_3, \ldots.$

The corresponding eigenvalues and eigenfunctions are

$\lambda_n = k_n^2, \quad y_n = \cos(k_n\pi) \quad n = 1, 2, 3, \ldots.$

Where k_n is the n-th positive root of $x = \cot x$.

Note this eigenvalue problem arises in a problem in quantum mechanics.