Question

Chebyshev's equation may be written

$$\frac{d}{dx}\left[(1-x^2)^{\frac{1}{2}}\frac{dy}{dx}\right]+n^2(1-x^2)^{-\frac{1}{2}}y=0$$

where n is a positive integer. The solutions which satisfy the boundary conditions $T_n(-1)=(-1)^n$ and $T_n(1)=1$ are polynomials of degree n. Find the first three Chebyshev polynomials. Prove that if $m\neq n$ then

$$\int_{-1}^{1}(1-x^2)^{-\frac{1}{2}}T_m(x)T_n(x)dx=0$$

Answer

To find the Chebyshev polynomials $T_n(x)$ one substitutes a general polynomial into the differential equation and uses the boundary conditions. We illustrate this with $T_2(x)$.

The general form of $T_2(x)$ is a quadratic so that $T_2(x)=a_0+a_1x+a_2x^2$. Using $T_2(1)=1$ gives $a_0+a_1+a_2=1$. Using $T_2(-1)=1$ gives $a_0-a_1+a_2=1$. Subtracting the equations gives $a_1=0$ and hence $a_0=1-a_2$. Writing a_2 as a we see that $T_2(x)=ax^2+(1-a)$.

We now substitute this into Chebyshev's equation (with $n=2$) and obtain $\{2a(1-2x^2)+4ax^2+4(1-a)\}(1-x^2)^{-1/2}=0, \quad \Rightarrow \quad 4-2a=0$. So that $a=2$ and $T_2(x)=2x^2-1$.

Now to show orthogonality. since T_n and T_m are solutions of Chebeyshev's equation we have:

$$\frac{d}{dt}\left[(1-x^2)\frac{dT_n}{dx}\right]+n^2(1-x^2)^{-1/2}T_n = 0 \quad (1)$$

$$\frac{d}{dt}\left[(1-x^2)\frac{dT_m}{dx}\right]+m^2(1-x^2)^{-1/2}T_m = 0 \quad (2)$$

Multiplying equation (1) by T_m and equation (2) by T_n, subtracting and integrating by parts between -1 and 1 gives:

$(n^2-m^2)\int_{-1}^{1}(1-x^2)^{-1/2}T_nT_m dx$

$=\int_{-1}^{1}\left\{T_n\left[\frac{d}{dx}(1-x^2)\frac{dT_m}{dx}\right]-T_m\left[\frac{d}{dx}(1-x^2)\frac{dT_n}{dx}\right]\right\}dx$

$=\left[(1-x^2)\left(T_n\frac{dT_m}{dx}-T_m\frac{dT_n}{dx}\right)\right]_{-1}^{1}$

$$-\int_{-1}^{1}\left\{(1-x^2)\frac{dT_n}{dx}\frac{dT_m}{dx}-(1-x^2)\frac{dT_m}{dx}\frac{dT_n}{dx}\right\}dx$$
$$=0$$
Since the first term vanishes at ± 1, and the terms and the integral cancel.