Question

Reduce each of the following equations to normal form and hence find their general solutions.

(a) $y'' + \frac{2}{x}y' + 4y = 0$

(b) $y'' + 2\sin x\, y' + \cos x(1 - \cos x)y = 0$

Answer

(a) $y'' + \dfrac{2}{x}y' + 4y = 0$. So $P(x) = \frac{2}{x}$ and $Q(x) = 4$.

We transform to a new variable $u(x)$ where $y(x) = u(x)v(x)$ and

$v(x) = \exp\left(-\dfrac{1}{2}\int P dx\right) = \exp\displaystyle\int -\dfrac{dx}{x} = \exp(-\ln x) = \dfrac{1}{x}$.

So $y(x) = \frac{u(x)}{x}$.

The equation for $u(x)$ is $u'' + q(x)u = 0$ where $q = Q - \frac{1}{4}P^2 - \frac{1}{2}P'$

Substituting for P and Q we find $q = 4 - \frac{1}{x^2} + \frac{1}{x^2} = 4$.

So that $u'' + 4u = 0, \quad \Rightarrow \quad u(x) = A\cos 2x + B\sin 2x$.

Hence $y(x) = \frac{1}{x}(A\cos 2x + B\sin 2x)$.

(b) $y'' + 2\sin xy' + \cos x(1 - \cos x)y$.

So $P(x) = 2\sin x$ and $Q(x) = \cos x(1 - \cos x)$.

We transform to a new variable $u(x)$ where $y(x) = u(x)v(x)$ and

$v(x) = \exp\left(-\dfrac{1}{2}\int P dx\right) = \exp\displaystyle\int -\sin x dx = e^{\cos x}$.

So $y(x) = e^{\cos x}u(x)$.

The equation for $u(x)$ is $u'' + q(x)u = 0$ where $q = Q - \frac{1}{4}P^2 - \frac{1}{2}P'$

Substituting for P and Q we find $q = \cos x - \cos^2 x - \sin^2 x - \cos x = -1$.

So that $u'' - u = 0, \quad \Rightarrow \quad u(x) = Ae^x + Be^{-x}$.

Hence $y(x) = e^{\cos x}(Ae^x + Be^{-x})$.