Exam Question
Topic: Fourier Series
Find the Fourier Series for the function given by

$$f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0; \\ x & \text{if } 0 \leq x \leq \pi. \end{cases}$$

Solution

$$a_0 = \frac{1}{\pi}\int_{-\pi} \pi f(x)\,dx = \frac{1}{\pi}\text{area of triangle} = \frac{1}{\pi}\frac{\pi^2}{2} = \frac{\pi}{2}.$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^\pi x\cos nx\,dx = \frac{1}{\pi}\left[x\frac{\sin nx}{n}\right]_0^\pi - \frac{1}{\pi}\int_0^\pi \frac{\sin nx}{n}\,dx \\ &= \frac{1}{\pi}\left[\frac{\cos nx}{n^2}\right]_0^\pi = \frac{1}{n^2\pi}\left[(-1)^n - 1\right] \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^\pi x\sin nx\,dx = \frac{1}{\pi}\left[-x\frac{\cos nx}{n}\right]_0^\pi + \frac{1}{\pi}\int_0^\pi \frac{\cos nx}{n}\,dx \\ &= \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n}\right] + \frac{1}{\pi}\left[\frac{\sin nx}{n^2}\right]_0^\pi = \frac{(-1)^{n+1}}{n} \end{aligned}$$

So the Fourier Series is

$$Frac\pi 4 + \sum_{n=1}^{\infty}\left[\frac{1}{\pi n^2}\left((-1)^n - 1\right)\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right]$$