QUESTION

Find the singularities of the following functions and work out the residues at these points.

(a) $\frac{1}{z^4+4z^2}$

(b) $\frac{1}{z^2 \sin z}$

ANSWER

(a) $$\frac{1}{z^4+4x^2} = \frac{1}{z^2(z^2+4)}.$$

This has a double pole at $z=0$ and simple poles at $z=\pm 2i$

Res(0)=$\lim_{z\to 0} \frac{d}{dz}\frac{1}{z^2+4} = \lim_{z\to 0} -\frac{2z}{(z^2+4)^2} = 0$

$\text{Res}(2i) = \lim_{z\to 2i} \frac{1}{z^2(z+2i)} = \frac{1}{-4\cdot 4i} = \frac{i}{16}$

$\text{Res}(-2i) = -\frac{i}{16}$

(b) $\frac{1}{z^2 \sin z} = \frac{1}{z^2(z-\frac{z^3}{3!}+\ldots)} = \frac{1}{z^3(1-\frac{z^2}{3!}+\ldots)} = z^{-3}(1+\frac{z^2}{3!}+\ldots) = z^{-3}+\frac{1}{3!}z^{-1}$

$z=0$ is a triple pole with residue $\frac{1}{3!}$.

$z=n\pi, n\in \mathbf{Z}$ are simple poles (except for $z=0$)

$\frac{1}{z^2 \sin z} \approx \frac{1}{(n\pi)^2(\cos n\pi)(z-n\pi)} \Rightarrow$ Res $= \frac{(-1)^n}{(n\pi)^2}$