QUESTION

Find the principal part of the Laurent series of the function

$$\frac{e^{zt}}{z^2\left(z^2+4z+5\right)}$$

about the point $z=0$.

Show that this Laurent series converges for $0<|z|<\sqrt{5}$.

ANSWER

Factorise: $z^2+4z+5=0 \Rightarrow (z+2)^2=4-5 \Rightarrow z=-2\pm i=z_\pm$

$$\frac{e^{zt}}{z^2\left(z-z_+\right)\left(z-z_-\right)}=z^{-2}e^{zt}\cdot\frac{1}{z+z_-}\cdot\frac{1}{1-\frac{z}{z_+}}\cdot\frac{1}{1-\frac{z}{z_-}}$$

We only need to expand to $O\left(z^{-1}\right)$, so

$$\begin{aligned}
&= z^{-2}\left(1+zt+\ldots\right)\frac{1}{z_+z_-}\left(1+\frac{z}{z_+}+\ldots\right)\left(1+\frac{z}{z_-}+\ldots\right)\\
&= \frac{1}{z_+z_-}z^{-2}\left[1+\left(t+\frac{1}{z_+}+\frac{1}{z_-}\right)z+\ldots\right]\\
&= \frac{1}{z_+z_-}+\left(\frac{t}{z_+z_-}+\frac{z_++z_-}{\left(z_+z_-\right)^2}\right)z^{-1}+\ldots\\
&= \frac{1}{3}z^{-2}+\left(\frac{t}{3}-\frac{4}{9}\right)z^{-1}+\ldots
\end{aligned}$$