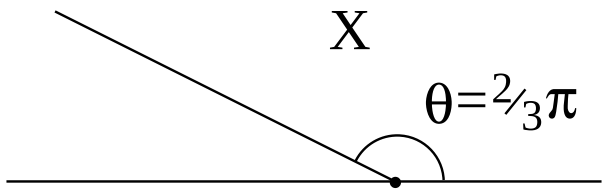
Question

Let X be the open wedge in $\mathbf{C}$ bounded by the positive real axis and the Euclidean ray from the origin making angle $\theta = \frac{2\pi}{3}$ with the positive real axis. Write down a bijective analytic map $\varphi : X \to \mathbf{H}$ with analytic inverse. Use this map φ to pull back the hyperbolic element of arc-length from $\mathbf{H}$ to X. (Give the hyperbolic element of arc-length on X explicitly as $\mu(z)\,|\mathrm{d}z|$.)

Write down the equations determining the hyperbolic lines in X.

Answer



$\phi : X \longrightarrow \mathbf{H}, \quad \phi(z) = z^{\frac{3}{2}}$

ϕ is bijective and has analytic inverse (and is analytic)

The element of arc-length on X is

$$ds_X = \frac{1}{\mathrm{Im}(\phi(\mathrm{z}))}|\phi'(z)||\,dz| = \frac{1}{\mathrm{Im}(\mathrm{z}^{\frac{3}{2}})}\frac{3}{2}|z^{\frac{1}{2}}||dz|$$

[Write $z = |z|e^{i\arg(z)}$ where $0 < \arg(\mathrm{z}) < \frac{2}{3}\pi$]

$$ds_X = \frac{3}{|z|\sin(\frac{3\arg(\mathrm{z})}{2})2}|\,dz|$$

$\underline{\omega = z^{\frac{3}{2}}}$ $\quad \underline{z = pe^{i\theta}}$

$\underline{\mathrm{Re}(\omega) = \mathrm{c}}$ $\quad p^{\frac{3}{2}}\cos\left(\frac{3\theta}{2}\right) = c$

$$|z^{\frac{3}{2}} - a|^2 = r^2$$

$\underline{|\omega - a|^2 = r^2}$ $\quad z^3 - az^{\frac{3}{2}} - a\bar{z}^{\frac{3}{2}} + a^2 = r^2$

$$\underline{z^3 - a^2\mathrm{Re}\left(\mathrm{z}^{\frac{3}{2}}\right) = \mathrm{r}^2 - \mathrm{a}^2}$$