Question

Prove that the identity map $g : \mathbf{H} \to \mathbf{H}$, defined by $g(x) = x$, gives a homeomorphism between the metric space $(\mathbf{H}, \mathrm{d}_{\mathbf{H}})$ and the metric space $(\mathbf{H}, f)$, where f is as defined in Problem 1 of this sheet.

Answer

Let $\begin{array}{rcl} \cup_\epsilon(x) & = & \{y \in \mathbf{H} | d_{\mathbf{H}}(x,y) < \epsilon\} \text{ and} \\ \cup_\epsilon^f(x) & = & \{y \in \mathbf{H} | f(x,y) < \epsilon\} \end{array}$

First note that g is a bijection, and so it remains only to show that $g : (\mathbf{H}, d_{\mathbf{H}}) \to (\mathbf{H}, f)$ and $g : (\mathbf{H}, f) \to (\mathbf{H}, d_{\mathbf{H}})$ are continuous (as $g^{-1} = g$).

$g : (\mathbf{H}, d_{\mathbf{H}}) \to (\mathbf{H}, f)$ is continuous at a if for every $\epsilon > 0$ there is $\delta > 0$ so that $g(\cup_{delta}(a)) \subseteq \cup_\epsilon^f(a)$; that is, that $\cup_{delta}(a) \subseteq \cup_\epsilon^f(a)$.

- If $d_{\mathbf{H}}(a,x) < \delta$, then $f(a,x) = \dfrac{d_{\mathbf{H}}(a,x)}{1 + d_{\mathbf{H}}(a,x)}$

$< d_{\mathbf{H}}(a,x) < \delta$,

so we may take $\delta = \epsilon$.

Going in the other direction, consider $g : (\mathbf{H}, f) \to (\mathbf{H}, d_{\mathbf{H}})$.

Given $\epsilon > 0$, choose $\delta = \dfrac{\epsilon}{1+\epsilon}$.

Then, if $f(a,x) = \dfrac{d_{\mathbf{H}}(a,x)}{1 + d_{\mathbf{H}}(a,x)} < \delta = \dfrac{\epsilon}{1+\epsilon}$ then $d_{\mathbf{H}}(a,x) < \epsilon$ as desired (since $g(t) = \dfrac{t}{1+t}$ is increasing in t).