Question
Consider the function function $f : \mathbf{H} \times \mathbf{H} \to \mathbf{R}$ defined by

$$f(x, y) = \frac{\mathrm{d}_{\mathbf{H}}(x, y)}{1 + \mathrm{d}_{\mathbf{H}}(x, y)}.$$

Prove that f is a metric on $\mathbf{H}$. Also, prove that the *diameter* of $\mathbf{H}$ with the metric f is finite, where the diameter $\mathrm{diam}(\mathbf{H}, f)$ of the metric space $(\mathbf{H}, f)$ is defined by

$$\mathrm{diam}(\mathbf{H}, f) = \sup\{f(x, y) \mid x,\ y \in \mathbf{H}\}.$$

Answer
Note that the first two conditions of a metric, that $f(x, y) \geq 0$ with equality if and only if $x = y$ and that $f(x, y) = f(y, x)$, are satisfied since they hold true for $d_{\mathbf{H}}(\cdot, \cdot)$. To check the triangle inequality, assume that it fails for $f(\cdot, \cdot)$, so that there are parts x, y, z, so that $f(x, y) > f(x, y) + f(y, z)$. Then,

$$\frac{d_{\mathbf{H}}(x, z)}{1 + d_{\mathbf{H}}(x, z)} > \frac{d_{\mathbf{H}}(x, y)}{1 + d_{\mathbf{H}}(x, y)} + \frac{d_{\mathbf{H}}(y, z)}{1 + d_{\mathbf{H}}(y, z)}$$

$d_{\mathbf{H}}(x, z)(1 + d_{\mathbf{H}}(x, y))(1 + d_{\mathbf{H}}(y, z)) >$
$d_{\mathbf{H}}(x, y)(1 + d_{\mathbf{H}}(x, z))(1 + d_{\mathbf{H}}(y, z)) + d_{\mathbf{H}}(y, z)(1 + d_{\mathbf{H}}(x, z))(1 + d_{\mathbf{H}}(x, y))$

Simplifying:

$d_{\mathbf{H}}(x, z) + d_{\mathbf{H}}(x, z)d_{\mathbf{H}}(x, y) + d_{\mathbf{H}}(x, z)d_{\mathbf{H}}(y, z) >$

$d_{\mathbf{H}}(x, y) + d_{\mathbf{H}}(x, y)d_{\mathbf{H}}(x, z) + d_{\mathbf{H}}(x, y)d_{\mathbf{H}}(y, z)$

$+d_{\mathbf{H}}(y, z) + d_{\mathbf{H}}(y, z)d_{\mathbf{H}}(x, z) + d_{\mathbf{H}}(y, z)d_{\mathbf{H}}(x, y)$

$+d_{\mathbf{H}}(x, y)d_{\mathbf{H}}(x, z)d_{\mathbf{H}}(y, z)$

Thus:

$d_{\mathbf{H}}(x, z) > d_{\mathbf{H}}(x, y) + d_{\mathbf{H}}(y, z) +$ stuff $> d_{\mathbf{H}}(x, y) + d_{\mathbf{H}}(y, z)$

But this is a contradiction, since $d_{\mathbf{H}}(\cdot, \cdot)$ is a metric. Hence, $f(\cdot, \cdot)$ satisfies the triangle inequality and hence is a metric.

$$\begin{aligned} \mathrm{diam}(\mathbf{H}, f) &= \sup\{f(x, y) | x, y \in \mathbf{H}\} \\ &= \sup\left\{ \frac{d_{\mathbf{H}}(x, y)}{1 + d_{\mathbf{H}}(x, y)} \middle| x, y \in \mathbf{H} \right\} \end{aligned}$$

Note that $\frac{d_{\mathbf{H}}(x,y)}{1+d_{\mathbf{H}}(x,y)} < 1$ for all $x, y \in \mathbf{H}$. Moreover,

$$f(i, \lambda i) = \frac{d_{\mathbf{H}}(i, \lambda i)}{1+d_{\mathbf{H}}(i, \lambda i)} = \frac{\ln(\lambda)}{1+\ln(\lambda)} \to 1 \text{ as } \lambda \to \infty,$$

and so $\operatorname{diam}(\mathbf{H}, f) = 1$.