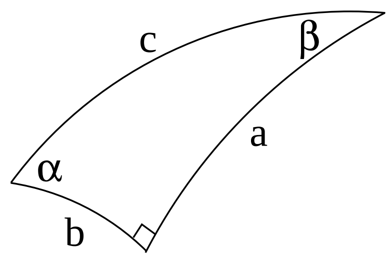
Question

Let T be a triangle with angles α, β, and $\frac{\pi}{2}$. Let a be the hyperbolic length of the side of T opposite the vertex with angle α, and let b be the hyperbolic length of the side of T opposite the vertex with angle β. Prove that $\tanh(b) = \sinh(a)\tan(\beta)$, that $\sinh(b) = \sinh(c)\sin(\beta)$, and that $\tanh(a) = \tanh(c)\cos(\beta)$.

Answer



- ls $\dfrac{\sinh(a)}{\sin(\alpha)} = \dfrac{\sinh(b)}{\sin(\beta)}$

  lcII $\cos(\beta) = -\cos(\alpha)\cos(\frac{\pi}{2}) + \sin(\alpha)\sin(\frac{\pi}{2})\cosh(b)$

  from ls: $\sinh(b) = \dfrac{\sinh(a)\sin(\beta)}{\sin(\alpha)}$

  use lcII: $\sinh(b) = \dfrac{\sinh(a)\sin(\beta)}{\cos(\beta)/\cosh(b)} = \sinh(a)\cosh(b)\tan(\beta)$

  So $\tanh(b) = \sinh(a)\tan(\beta)$ as desired. $(\star)$

- $\sinh(b) = \sinh(c)\sin(\beta) = \dfrac{\sinh(c)}{\sin(\beta)}\sin(\beta)$ $\left(\text{with } \gamma = \dfrac{\pi}{2}\right)$

  immediately from ls.

- from above $(\star)$: $\tanh(b) = \sinh(a)\tan(\beta)$

  from ls and lcII:

  $\cosh(c) = \cot(\alpha)\cot(\beta)$

  (with $\gamma = \dfrac{\pi}{2}$ and using

  $\cos(\gamma) = -\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)\cosh(c)$)

  $\sinh(c) = \dfrac{\sinh(a)}{\sinh(\alpha)}$

$$\begin{aligned} \tanh(c) &= \frac{\sinh(a)}{\sin(\alpha)} \cdot \tan(\alpha)\tan(\beta) \\ &= \sinh(a) \cdot \tan(\beta) \cdot \frac{1}{\cos(\alpha)} \end{aligned}$$

and so

$$\begin{aligned} \tanh(c)\cos(\alpha) &= \sinh(a)\tanh(\beta) \\ &= \tanh(b) \end{aligned}$$

(from $(\star)$) (as desired)