Question

Show that the following formula holds in $\mathbf{H}$:

$$\cosh(\mathrm{d}_{\mathbf{H}}(z,w)) = 1 + \frac{|z-w|^2}{2\mathrm{Im}(z)\mathrm{Im}(w)}.$$

Answer

Start by showing that the RHSide of the equation is invariant under $\text{Möb}^+(\mathbf{H})$: that is, if $m \in \text{Möb}^+(\mathbf{H})$, then

$$\frac{|z-w|^2}{2\mathrm{Im}(z)\mathrm{Im}(w)} = \frac{|m(z)-m(w)|^2}{2\mathrm{Im}(m(z))\mathrm{Im}(m(w))} : (\star)$$

Write $m(z) = \dfrac{az+b}{cz+d}$ $a,b,c,d \in \mathbf{R},\ ad-bc=1$

Then:

(1)

$$\begin{aligned}|m(z)-m(w)|^2 &= |m'(z)||m'(w)||z-w|^2 \\ &= \frac{|z-w|^2}{|cz+d|^2|cw+d|^2}\end{aligned}$$

$$\begin{aligned}m(z)-m(w) &= \frac{az+b}{cz+d} - \frac{aw+b}{cw+d} \\ &= \frac{(az+b)(cw+d)-(aw+b)(cz+d)}{(cz+d)(cw+d)} \\ &= \frac{z-w}{(cz+d)(cw+d)}\end{aligned}$$

$$m'(z) = \frac{1}{(cz+d)^2}$$

(2)

$$\begin{aligned}\mathrm{Im}(m(z)) &= \mathrm{Im}\left(\frac{az+b}{cz+d}\cdot\frac{c\bar{z}+d}{c\bar{z}+d}\right) \\ &= \mathrm{Im}\left(\frac{acz\bar{z}+bc\bar{z}+adz+bd}{(cz+d)(c\bar{z}+d)}\right) \\ &= \frac{\mathrm{Im}(z)}{(cz+d)(c\bar{z}+d)}\end{aligned}$$

and we see that $(\star)$ is satisfied.

Now, given $z, w \in \mathbf{H}$, choose $m \in \text{Möb}^+(\mathbf{H})$ so that

$(m(z) = i$ and $m(w) = \lambda i (\lambda > 1)$, where $\ln(\lambda) = d_{\mathbf{H}}(z, w)$: then

- $\cosh(d_{bfH}(z, w)) = \cosh(\ln(\lambda)) = \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right) = \frac{\lambda^2+1}{2\lambda}$
- $$\begin{aligned} 1 + \frac{|z-w|^2}{2\text{Im}(z)\text{Im}(w)} &= 1 + \frac{|i - \lambda i|^2}{2 \cdot 1 \cdot \lambda} \\ &= 1 + \frac{(\lambda-1)^2}{2\lambda} \\ &= \frac{2\lambda + \lambda^2 - 2\lambda + 1}{2\lambda} \\ &= \frac{\lambda^2+1}{2\lambda} \text{ as desired} \end{aligned}$$