Question

This question introduces an example where the value of the integral depends on the order of integration. Show that:

(a)

$$\int_{x=0}^{x=1}\int_{y=0}^{y=1}\frac{x-y}{(x+y)^3}\,dy\,dx=\frac{1}{2}$$

(b)

$$\int_{y=0}^{y=1}\int_{x=0}^{x=1}\frac{x-y}{(x+y)^3}\,dy\,dx=\frac{1}{2}$$

What feature of this function may be causing the answers to be different? (**HINT:** In part (a) evaluate the integral by expressing the integrand using the method of partial fractions. In part (b) consider the effect on the integral of formally interchanging the variables x and y.)

Answer

(a)

$$\frac{x-y}{(x+y)^3}=\frac{2x-x-y}{(x+y)^3}=\frac{2x}{(x+y)^3}-\frac{x+y}{(x+y)^3}=\frac{2x}{(x+y)^3}-\frac{1}{(x+y)^2}$$

The integral becomes:

$$\int_{x=0}^{x=1}\left\{\int_{y=0}^{y=1}\frac{2x}{(x+y)^3}-\frac{1}{(x+y)^2}\,dy\right\}\,dx$$

$$\begin{aligned}
&= \int_{x=0}^{x=1}\left[\frac{-x}{(x+y)^2}+\frac{1}{x+y}\right]_{y=0}^{y=1}\,dx\\
&= \int_{x=0}^{x=1}\frac{-x}{(x+1)^2}+\frac{1}{x+1}+\frac{1}{x}-\frac{1}{x}\,dx\\
&= \int_{x=0}^{x=1}\frac{1}{x+1}-\frac{x}{(x+1)^2}\,dx\\
&= \int_{x=0}^{x=1}\frac{1}{(x+1)^2}\,dx=\left[\frac{-1}{x+1}\right]_0^1=-\frac{1}{2}+1=\frac{1}{2}
\end{aligned}$$

(b) Formally interchanging x and y gives

$$\int_{y=0}^{y=1}\int_{x=0}^{x=1}\frac{y-x}{(y+x)^3}\,dx\,dy=\frac{1}{2}$$

So that

$$\int_{y=0}^{y=1}\int_{x=0}^{x=1}\frac{x-y}{(x+y)^3}\,dx\,dy=-\int_{y=0}^{y=1}\int_{x=0}^{x=1}\frac{y-x}{(y+x)^3}\,dx\,dy=-\frac{1}{2}$$

The difference arises because $\frac{x-y}{(x+y)^3}$ has a singularity at $(x,y)=(0,0)$ [hence the function is not properly defined at the point $(0,0)$].