Question

Evaluate the following double integral (you should first sketch the region of integration):

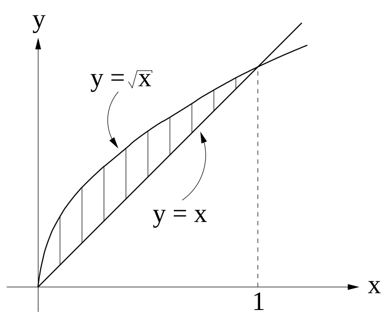
$$\int_{x=0}^{x=1} \int_{y=x}^{y=\sqrt{x}} x^2 y \, dy dx.$$

Reverse the order of integration and show that this gives the same answer.

Answer

The region of integration is defined as follows:

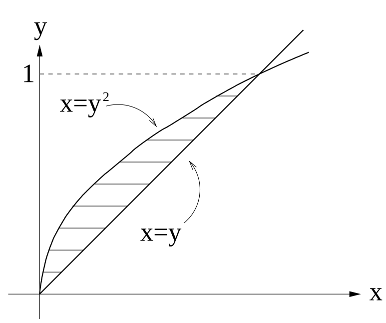
$$\begin{aligned} \text{first keep x constant and vary y} : &\quad x \le y \le \sqrt{x} \\ \text{then vary x} : &\quad 0 \le x \le 1 \end{aligned}$$



(Note : y = $\sqrt{\text{x}}$is one branch of the parabola $x = y^2$)

$$\begin{aligned} \int_{x=0}^{x=1} \left\{ \int_{y=x}^{y=\sqrt{x}} x^2 y \, dy \right\} dx &= \int_0^1 \left[\frac{x^2 y^2}{2} \right]_{y=x}^{y=\sqrt{x}} dx \\ &= \int_0^1 \frac{x^3}{2} - \frac{x^4}{2} \, dx \\ &= \left[\frac{x^4}{8} - \frac{x^5}{10} \right]_0^1 \\ &= \frac{1}{8} - \frac{1}{10} \\ &= \frac{1}{40} \end{aligned}$$

To reverse the order of integration, take horizontal lines (y fixed):



$$\begin{aligned} \text{first vary } x &: y^2 \le x \le y \\ \text{then vary } y &: 0 \le y \le 1 \end{aligned}$$

Integral becomes:

$$\begin{aligned}
\int_{y=0}^{y=1} \left\{ \int_{x=y^2}^{x=y} x^2 y \, dx \right\} dy &= \int_0^1 \left[\frac{x^3 y}{3} \right]_{x=y}^{x=y^2} dy \\
&= \int_0^1 \frac{y^4}{3} - \frac{y^7}{3} dy \\
&= \left[\frac{y^5}{15} - \frac{y^8}{24} \right]_0^1 \\
&= \frac{1}{15} - \frac{1}{24} \\
&= \frac{3}{120} \\
&= \frac{1}{40}
\end{aligned}$$