QUESTION
Prove

(a) using the factorial definition,

(b) using the word definition

(i) $\begin{pmatrix} n \\ n-r \end{pmatrix} = \begin{pmatrix} n \\ r \end{pmatrix}$

(ii) $\begin{pmatrix} n+1 \\ r \end{pmatrix} = \begin{pmatrix} n \\ r \end{pmatrix} + \begin{pmatrix} n \\ r-1 \end{pmatrix}$

ANSWER
(i) (a) $\begin{pmatrix} n \\ n-r \end{pmatrix} = \dfrac{n!}{(n-r)!(n-(n-r))!} = \dfrac{n!}{(n-r)!r!} = \begin{pmatrix} n \\ r \end{pmatrix}$
(b) $\begin{pmatrix} n \\ r \end{pmatrix}$ is the number of ways of choosing r from n, each choice of r is effectively one different choice of $n-r$ because $n-r$ are left behind.
(ii) (a)

$$\begin{aligned} \begin{pmatrix} n \\ r \end{pmatrix} + \begin{pmatrix} n \\ r-1 \end{pmatrix} &= \frac{n!}{(n-r)!r!} + \frac{n!}{(n-r+1)!(r-1)!} \\ &= \frac{n!}{(n-r+1)!r!}\{n-r+1+r\} \\ &= \frac{(n+1)!}{(n+1-r)!r!} = \begin{pmatrix} n+1 \\ r \end{pmatrix} \end{aligned}$$

(b) Consider $n+1$ things. $\underbrace{\times \times \times \times \ldots \times}_{n} \underbrace{\text{O}}_{1}$
Choosing r from $n+1$ is $\begin{pmatrix} n+1 \\ r \end{pmatrix}$. Each choice either included O in which case you need to choose $r-1$ from n, $\begin{pmatrix} n \\ r-1 \end{pmatrix}$, or excludes O, in which case we need to choose r from n, $\begin{pmatrix} n \\ r \end{pmatrix}$. Hence

$$\begin{pmatrix} n+1 \\ r \end{pmatrix} = \begin{pmatrix} n \\ r \end{pmatrix} \begin{pmatrix} n \\ r-1 \end{pmatrix}$$