Question
Prove, if f is continuous and if $\lim_{x\to\infty}(f(x+1)-f(x))=0$, that

$$\lim_{x\to\infty}\frac{f(x)}{x}=0.$$

Answer
First, since $\lim_{x\to\infty}(f(x+1)-f(x))=0$, for any $\varepsilon>0$, there exists x_0 (which we can take to be positive) so that $|f(x+1)-f(x)|<\frac{1}{2}\varepsilon$ for $x>x_0$. Now, using the maximum value property (see note below), there exists a maximum value M of $|f(x)|$ on the interval $[x_0,x_0+1]$.

The first claim is that for any $k\geq 0$, we have that $|f(x)|\leq\frac{k}{2}k\varepsilon+M$ for x in the interval $[x_0+k,x_0+k+1]$. To see this, let K be the maximum value of $|f(x)|$ on $[x_0+k,x_0+k+1]$, occurring at y. Then, $x_0+k\leq y\leq x_0+k+1$, and so $x_0\leq y-k\leq x_0+1$. We now engage in some algebraic manipulation:

$$\begin{aligned}
|f(y)| &= |f(y)-f(y-k)+f(y-k)| \\
&\leq |f(y)-f(y-k)|+|f(y-k)| \\
&\leq |f(y)-f(y-1)+f(y-1)-\cdots f(y-k+1)+f(y-k+1)-f(y-k)|+|f(y-k)| \\
&\leq |f(y)-f(y-1)|+|f(y-1)-f(y-2)|+\cdots+|f(y-k+1)-f(y-k)|+|f(y-k)| \\
&\leq \frac{1}{2}\varepsilon+\frac{1}{2}\varepsilon+\cdots+\frac{1}{2}\varepsilon+M \\
&\leq \frac{k}{2}\varepsilon+M.
\end{aligned}$$

In particular, this tells us that

$$\frac{|f(y)|}{y}\leq\frac{\frac{k}{2}\varepsilon+M}{y}\leq\frac{\frac{k}{2}\varepsilon}{y}+\frac{M}{y}\leq\frac{\frac{k}{2}\varepsilon}{x_0+k}+\frac{M}{y}<\frac{\frac{k}{2}\varepsilon}{k}+\frac{M}{y}<\frac{1}{2}\varepsilon+\frac{M}{y}$$

for all y in the interval $[x_0+k,x_0+k+1]$.

Now, choose $x_1>x_0$ so that $\frac{M}{x_1}<\frac{1}{2}\varepsilon$. Then, for all $y>x_1$, we have that

$$\left|\frac{f(y)}{y}\right|=\frac{|f(y)|}{y}<\frac{1}{2}\varepsilon+\frac{M}{y}<\frac{1}{2}\varepsilon+\frac{1}{2}\varepsilon$$

for all $y>x_1$. In particular, we have that the definition of $\lim_{x\to\infty}\frac{f(x)}{x}=0$ is satisfied, as desired.
Note:
Maximum value property for continuous functions: Let f be a function that is continuous on the closed interval $[a,b]$. Then f achieves its maximum on $[a,b]$; that is, there exists some x_0 in $[a,b]$ so that $f(x_0)\geq f(x)$ for all $x\in[a,b]$.