Question

Prove, using the definition, that each of the following functions is continuous at all points of $\mathbf{R}$.

1. $h_n(x) = x^n$, where $n \in \mathbf{N}$;
2. $g(x) = c$, where $c \in \mathbf{R}$;
3. f is a function on $\mathbf{R}$ which satisfies $|f(x) - f(y)| \le c|x - y|$ for all x, $y \in \mathbf{R}$, where $c > 0$ is a constant.

Answer

1. To show that $h_n(x)$ is continuous at $a \in \mathbf{R}$, we need to show that $\lim_{x\to a} h_n(x) = h_n(a)$. Recalling the definition of limit, this translates to showing that for each $\varepsilon > 0$, there exists $\delta > 0$ so that if $|x - a| < \delta$, then $|h_n(x) - h_n(a)| < \varepsilon$. Since $h_n(x) = x^n$, this is the same as showing that for each $\varepsilon > 0$, there exists $\delta > 0$ so that if $|x - a| < \delta$, then $|x^n - a^n| < \varepsilon$. Let's break the proof into cases.

   If $n = 1$, then all we need to do to satisfy the definition is take $\delta = \varepsilon$. So, we can assume that $n \ge 2$. If in addition we have that $a = 0$, then by the definition of limit, we need to show that for each $\varepsilon > 0$, there is $\delta > 0$ so that if $|x| < \delta$, then $|x^n| = |x|^n < \varepsilon$. So, taking $\delta = \varepsilon^{1/n}$, we are done in this case as well.

   Consider now the case that $n \ge 2$ and $a > 0$, and factor $|x^n - a^n|$ to get $|x^n - a^n| = |(x-a)(x^{n-1} + ax^{n-2} + \cdots + a^{n-2}x + a^{n-1})|$. Recall that we have a great deal of choice in how we choose δ, so we may restrict our attention to the interval $|x - a| < \frac{1}{2}a$, so that $\frac{1}{2}a < x < \frac{3}{2}a$, by requiring that $\delta < \frac{1}{2}a$ (which makes sense, since $a > 0$). Calculating, we see that

$$\begin{aligned}
|x^n - a^n| &= |(x-a)(x^{n-1} + ax^{n-2} + \cdots + a^{n-2}x + a^{n-1})| \\
&\le |x-a|(x^{n-1} + ax^{n-2} + \cdots + a^{n-2}x + a^{n-1}) \\
&< |x-a|\left(\left(\frac{3}{2}a\right)^{n-1} + a\left(\frac{3}{2}a\right)^{n-2} + \cdots + a^{n-2}\frac{3}{2}a + a^{n-1}\right) \\
&= |x-a|a^{n-1}\sum_{k=0}^{n-1}\left(\frac{3}{2}\right)^k \\
&= |x-a|a^{n-1}\frac{1-(3/2)^n}{1-(3/2)} = C|x-a|,
\end{aligned}$$

where $C = a^{n-1}\frac{1-(3/2)^n}{1-(3/2)} > 0$ depends on both $a > 0$ and $n \geq 2$. So, take δ to be the smaller of $\frac{1}{C}\varepsilon$ and $\frac{1}{2}a$. Then, for $|x - a| < \delta$, we have that $|x^n - a^n| < C|x - a| \leq \varepsilon$ as desired. (The first inequality follows from the calculation above and the fact that $|x - a| < \delta < \frac{1}{2}a$, while the second inequality follows from $\delta < \frac{1}{C}\varepsilon$.)

A similar argument, with appropriate placements of absolute values, holds for $a < 0$. (Note that for a given $\varepsilon > 0$, the choice of δ depends on ε, on a, and on n.)

2. To show that $g(x)$ is continuous at $a \in \mathbf{R}$, we need to show that $\lim_{x\to a} g(x) = g(a)$. Recalling the definition of limit, this translates to showing that for each $\varepsilon > 0$, there exists $\delta > 0$ so that if $|x - a| < \delta$, then $|g(x) - g(a)| < \varepsilon$. Since $g(x) = c$ for all x, this is the same as showing that for each $\varepsilon > 0$, there exists $\delta > 0$ so that if $|x - a| < \delta$, then $|c - c| = 0 < \varepsilon$. So, regardless of the value of ε, taking $\delta = 1$ (or whatever your favorite positive number happens to be today) satisfies the definition.

3. To show that $f(x)$ is continuous at $a \in \mathbf{R}$, we need to show that $\lim_{x\to a} f(x) = f(a)$. Recalling the definition of limit, this translates to showing that for each $\varepsilon > 0$, there exists $\delta > 0$ so that if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$. Since $|f(x) - f(a)| \leq c|x - a|$, taking $\delta = \frac{1}{c}\varepsilon$ satisfies the definition. (If $|x - a| < \delta = \frac{1}{c}\varepsilon$, then $|f(x) - f(a)| \leq c|x - a| < c\frac{1}{c}\varepsilon = \varepsilon$, as desired.) (Functions that satisfy this condition are often referred to as **Lipschitz functions**.)