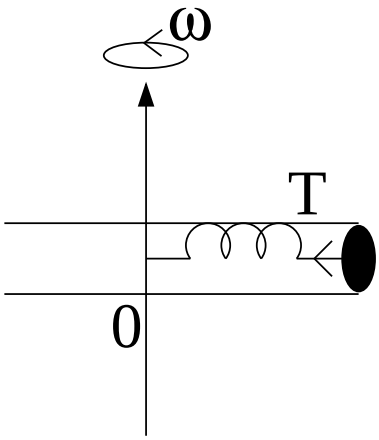
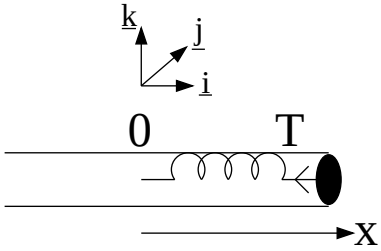
Question

A smooth straight tube rotates in a plane with constant angular velocity ω about a perpendicular axis through a point O. Inside the tube is a particle of mass m joined to O by a spring of stiffness $k = 5m\omega^2$ and natural length a. The particle is released from its position of relative rest, with the spring at its natural length. Show the the particle describes oscillations with period π/ω and amplitude $a/4$ relative to the tube. What is the largest reaction on the tube?

Answer



Newton's 2nd law: In a frame of reference stationary with respect to the tube:



gives $m\ddot{x}\mathbf{i} = -T\mathbf{i} - mg\mathbf{k} + \mathbf{R} - \underbrace{2m\boldsymbol{\omega} \times \mathbf{v}}_{\text{Coriolis}} - \underbrace{m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})}_{\text{Centrifugal}}$

$\mathbf{v} = \mathbf{i}\dot{x}$ and $\mathbf{R}$ is perpendicular to $\mathbf{i}$ as the tube is smooth.
$\boldsymbol{\omega} = \omega\mathbf{k}$, therefore

$$m\ddot{x} = -5m\omega^2(x - a) + m\omega^2 x$$

$$
\begin{aligned}
\ddot{x} &= -4\omega^2 x + 5\omega^2 a \\
&= -4\omega\left(x - \frac{5}{4}a\right) \\
x - \frac{5}{4}a &= A\cos(2\omega t + \alpha) \quad A, \alpha \text{ are constant} \\
&\quad \text{at } t = 0,\ x = a,\ \dot{x} = 0 \Rightarrow A = -\frac{a}{4}\alpha = 0 \\
x &= a\left[\frac{5}{4} - \frac{1}{4}\cos 2\omega t\right]
\end{aligned}
$$

The period of oscillation is $\dfrac{2\pi}{2\omega} = \dfrac{\pi}{\omega}$
The amplitude of oscillation is $\dfrac{a}{4}$
$\mathbf{R} = mg\mathbf{k} + 2m\boldsymbol{\omega} \times \mathbf{v} = mg\mathbf{k} + 2m\omega\frac{a}{2}\omega \sin 2\omega t\mathbf{j}$
Therefore $R_{\max} = m\sqrt{g^2 + a^2\omega^4}$