Question

Show that the small angular deviation ϵ of a plumb line from the direction to the centre of the earth at a point on the earth's surface at a latitude λ is

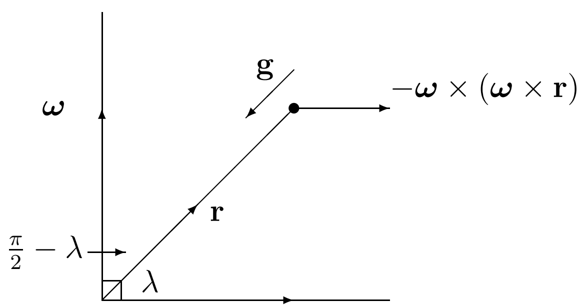
$$\epsilon = \frac{Rw^2 \sin\lambda \cos\lambda}{g_0 - Rw^2 \cos^2\lambda}$$

where R is the radius of the earth. What is the value of the maximum deviation and at what latitude does this occur?

Answer

$\mathbf{g} = \mathbf{g}_0 - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \underline{r})$

$\mathbf{g} = \mathbf{g}_0 - [(\boldsymbol{\omega} \cdot \mathbf{r})\boldsymbol{\omega} - \omega^2 \mathbf{r}] \quad (*)$



Taking the cross product with $\mathbf{g}_0$

$$\begin{aligned}
\mathbf{g} \times \mathbf{g}_0 &= \mathbf{g}_0 \times \mathbf{g_0} + \omega^{\mathbf{2}}\mathbf{r} \times \mathbf{g_0} - (\boldsymbol{\omega} \cdot \mathbf{r})\boldsymbol{\omega} \times \mathbf{g_0} \\
&= -(\underline{\omega} \cdot \mathbf{r})\boldsymbol{\omega} \times \mathbf{g}_0 \qquad \text{as } \mathbf{r}, \mathbf{g}_0 \text{ are parallel}
\end{aligned}$$

$$\begin{aligned}
g g_0 \sin\epsilon &= \omega R \cos\left(\frac{\pi}{2} - \lambda\right) \omega g_0 \sin\left(\frac{\pi}{2} - \lambda\right) \qquad \text{by defn of } \epsilon \\
\sin\epsilon &= \frac{\omega^2 R}{g} \sin\lambda \cos\lambda
\end{aligned}$$

Take the dot product of (*) with itself to find g:

$$\begin{aligned}
g^2 &= g_0^2 - 2\mathbf{g}_0 \cdot [\omega^2 \mathbf{r} - (\boldsymbol{\omega} \cdot \mathbf{r})\boldsymbol{\omega}] + \text{terms including } \omega^{\mathbf{4}} \\
&\approx g_0^2 - 2g_2[\omega^2 R - \omega R \sin\lambda \omega \sin\lambda]
\end{aligned}$$

$$
\begin{aligned}
&\approx g_0^2 - 2g_0\omega^2 R\cos^2\lambda \\
g &\approx g_0\left(1 - 2\frac{\omega^2 R}{g_0}\cos^2\lambda\right)^{\frac{1}{2}} \\
&\approx g_0 + \omega^2 R\cos^2\lambda
\end{aligned}
$$

Thus $\sin\epsilon \approx \dfrac{R\omega^2\sin\lambda\cos\lambda}{g_0 - R\omega^2\cos^2\lambda}$
ϵ is clearly small so $\sin\epsilon \approx \epsilon$, hence the result.