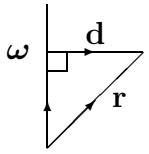
Question

A particle with position vector $\mathbf{r}$ relative to an origin O, rotates with angular velocity $\boldsymbol{\omega}$ about an axis through O. Show that the magnitude of the centripetal acceleration of the particle, $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$, is $\omega^2 d$, where d is the perpendicular distance of the particle from the axis of rotation.

Answer



Now $\mathbf{r} = \lambda\boldsymbol{\omega} + \mathbf{d}$ where $\boldsymbol{\omega}, \mathbf{d}$ are perpendicular. $\Rightarrow \boldsymbol{\omega} \cdot \mathbf{d} = 0$
Therefore $\mathbf{r} \cdot \boldsymbol{\omega} = \lambda\omega^2 + \mathbf{d} \cdot \boldsymbol{\omega} = \lambda\omega^2$
Therefore $\mathbf{r} = \dfrac{1}{\omega^2}(\mathbf{r} \cdot \boldsymbol{\omega})\boldsymbol{\omega} + \mathbf{d}$

$$
\begin{aligned}
\text{Centripetal acceleration} &= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \\
&= \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \left(\frac{1}{\omega^2}(\mathbf{r} \cdot \boldsymbol{\omega})\boldsymbol{\omega} + \mathbf{d}\right)\right) \\
&= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{d}) \qquad \text{as } \boldsymbol{\omega} \times \boldsymbol{\omega} = 0 \\
&= (\boldsymbol{\omega} \cdot \mathbf{d})\boldsymbol{\omega} - \boldsymbol{\omega} \cdot \boldsymbol{\omega}\mathbf{d} \qquad \text{as } \boldsymbol{\omega} \cdot \mathbf{d} = 0 \\
&= -\omega^2\mathbf{d} \qquad \text{as required}
\end{aligned}
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