QUESTION

Expand the following functions in Taylor or Laurent series about each of the points $z = 0$ and $z = i$

(a) $\frac{1}{1+z}$

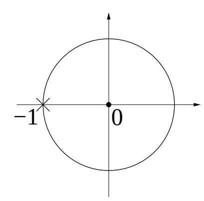
(b) $\frac{1}{z^2(z-i)}$

(c) $\frac{\cosh z}{1+z^2}$

ANSWER

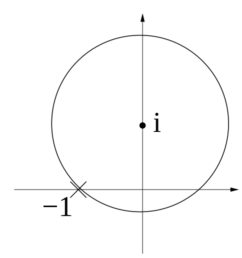
(a) (i)

$$\frac{1}{1+z} = \frac{1}{1-(-z)} = 1 + (-z) + (-z)^2 + \ldots$$
$$= 1 - z + z^2 - z^3 + \ldots$$
for $|z| < 1$.



(ii)

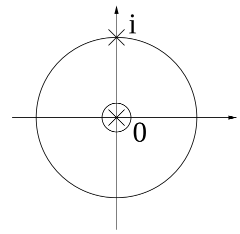
$$\frac{1}{1+z} = \frac{1}{1+i+(z-1)}$$
$$\frac{1}{1+i}\frac{1}{1+\frac{z-i}{1+i}}$$
$$\frac{1}{1+i}(1 - \frac{z-i}{1+i} + (\frac{z-i}{1+i})^2 - (\frac{z-i}{1+i})^3 + \ldots)$$
for $|\frac{z-i}{1+i}| < 1 \Longleftrightarrow |z-i| \leq \sqrt{2}$



(b) (i)

$$\begin{aligned}
\frac{1}{z^2(z-i)} &= \frac{1}{z^2}\frac{1}{-i}\frac{1}{1+\frac{z}{-i}} \\
&= \frac{i}{z^2}\frac{1}{1+iz} \\
&= \frac{i}{z^2}[1 - iz + (iz)^2 - (iz)^3 + (iz)^4 - \ldots] \\
&= \frac{i}{z^2}[1 - iz - z^2 + iz^3 + z^4 - \ldots] \\
&= iz^{-2} + z^{-1} - i - z + iz^2 + \ldots
\end{aligned}$$
for $0 < |z| < 1$

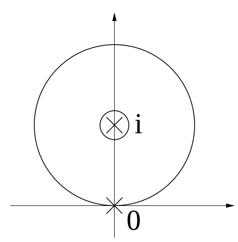
The principal part is $iz^{-2} + z^{-1}$
Res $= 1$.



(ii)

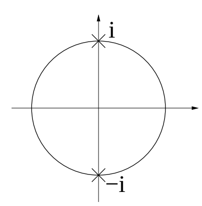
$$
\begin{aligned}
\frac{1}{z^2(z-i)} &= \frac{1}{z-i}\left[\frac{1}{(z-i)+i}\right]^2 \\
&= \frac{1}{z-i}\left[\frac{1}{i}\frac{1}{1+\frac{z-i}{i}}\right]^2 \quad \text{for } 0 < |z-i| < 1 \\
&= -\frac{1}{z-i}\left[1 - \frac{z-i}{i} + \left(\frac{z-i}{i}\right)^2 - \left(\frac{z-i}{i}\right)^3 + \ldots\right]^2 \\
&= -\frac{1}{z-i}\left[1 - 2\frac{z-i}{i} + \ldots\right] \\
&= -(z-i)^{-1} - 2i + \ldots
\end{aligned}
$$

Res$=-1$



(c) (i)

$\dfrac{\cosh z}{1+z^2}$
$= (1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots)(1 - z^2 + z^4 - \ldots)$

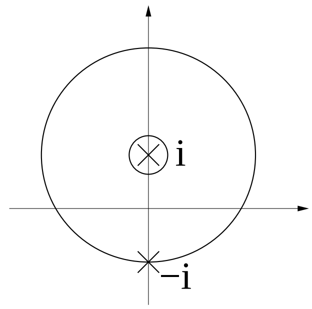


This is the Taylor series of cosh (valid for all x) times the geometric series in $(-z^2)$ (valid for $|z^2| < 1 \Longleftrightarrow |z| < 1$.) Hence

$$\frac{\cosh z}{1+z^2} = 1 + (\frac{1}{2!} - 1)z^2 + (-1 \cdot \frac{1}{2!} + 1 + \frac{1}{4!})z^4 + \ldots \text{ for } |z| < 1.$$

(ii)

$$\frac{\cosh z}{1+z^2} = \frac{\cosh z}{(z+i)(z-i)}$$



$$
\begin{aligned}
&= \frac{1}{z-i}\frac{1}{(z-i)+2i}\cosh z \\
&= \frac{1}{z-i}\frac{1}{2i}\frac{1}{1+\frac{z-i}{2i}}\cosh z \\
&= \frac{1}{z-i}\frac{1}{2i}\left[\text{ geometric series in } \frac{z-i}{2i} \text{ for } |z-i|<2\right] \\
&\quad [\text{ Taylor series of } \cosh z \text{ about } z=i, \text{ for all } z] \\
&= \frac{1}{z-i}\frac{1}{2i}\left[1-\frac{z-i}{2i}+(\frac{z-i}{2i})^2-\dots\right] \\
&= \left[\cosh(i)+\sinh(i)(z-i)+\frac{\cosh(i)}{2!}(z-i)^2+\dots\right] \\
&= \frac{1}{2i}\frac{1}{z-i}\left[1-\frac{z-i}{2i}+\dots\right][\cos(1)+i\sin(1)(z-i)+\dots] \\
&= \frac{1}{2i}\left[\cos(1)(z-i)^{-1}+(-\frac{1}{2i}+i\sin(1))+\dots\right]
\end{aligned}
$$

(Using $\cosh(ix)=\cos(x),\ \sinh(ix)=i\sin(x)$)

Res= $\frac{\cos(1)}{2i}$ This is a Laurent series for $0<|z-i|<2$