QUESTION

Evaluate the following directly by parameterising the curves:

(a) $\int_\gamma |z|\, dz$

where γ is

(i) the straight line joining $z = -1 - i$ to $z = 1 + i$,

(ii) the upper unit semicircle centred on O, joining $z = 1$ to $z = -1$,

(iii) the lower unit semicircle centred on O, joining $z = -1$ to $z = 1$.

(b) $\int_\gamma z^{\frac{1}{2}}\, dz$

where γ is

(i) the straight line joining $z = -1$ to $z = i$,

(ii) the left unit semicircle centered on O, joining $z = i$ to $z = -i$.

(c) $\int_\gamma \sinh z\, dz$

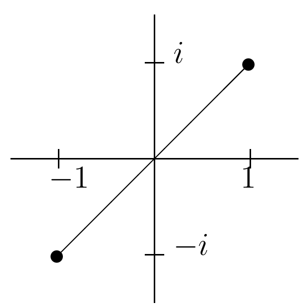
where γ is the straight line joining $z = 0$ to $z = i$.

ANSWER

(a) **(i)**

$z(t) = (1+i)t, \quad -1 \le t \le 1$

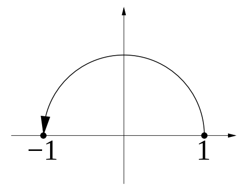
$|z(t)| = \sqrt{2}|t| \quad \frac{dz}{dt} = 1 + i$



$$\begin{aligned}
\int_\gamma |z|\, dz &= \int_{-1}^{1} \sqrt{2}|t|(1+i) \\
&= 2\int_0^1 \sqrt{2}t(1+i)\, dt \\
&= 2\sqrt{2}(1+i)\left[\frac{1}{2}t^2\right]_0^1 \\
&= (1+i)\sqrt{2}
\end{aligned}$$

(ii)

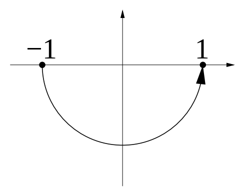
$z(t) = e^{it}, \quad 0 \leq t \leq \pi$
$|z(t)| = 1 \quad \frac{dz}{dt} = ie^{it}$



$$\int_\gamma |z|\, dz = \int_0^\pi ie^{it}\, dt = \left[e^{it}\right]_0^\pi = -2$$

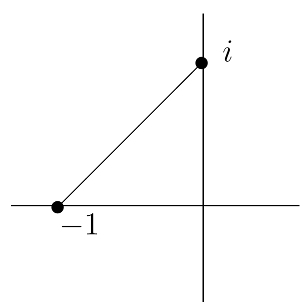
(iii)

$z(t) = e^{it},$
$\pi \leq t \leq 2\pi$
$\int_\gamma |z|\, dz = [e^{it}]_\pi^{2\pi} = 2$



(b) (i)

$z(t) = (1+i)t - 1, \quad 0 \leq t \leq 1$
$\frac{dz}{dt} = 1 + i$

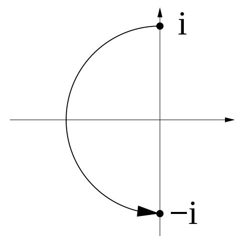


$$\begin{aligned}\int_\gamma z^{\frac{1}{2}}\, dz &= \int_0^1 \left[(1+i)t - 1\right]^{\frac{1}{2}} (1+i)\, dt \\ &= \left[\frac{2}{3}((1+i)t-1)^{\frac{3}{2}}\right]\end{aligned}$$

$$= \frac{2}{3}\left[(i)^{\frac{3}{2}} - (-1)^{\frac{3}{2}}\right]$$
$$= \frac{2}{3}\left(e^{\frac{3}{2}\frac{i\pi}{2}} - e^{\frac{3}{2}i\pi}\right)$$
$$= \frac{2}{3}\left[\frac{1}{\sqrt{2}}(-1+i) - (-i)\right]$$
$$= \frac{2}{3}\left[-\frac{1}{\sqrt{2}} + \left(1 + \frac{1}{\sqrt{2}}\right) i\right]$$

(ii)

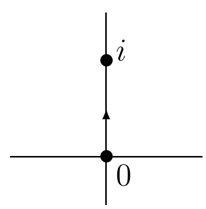
$z(t) = e^{it}, \quad \frac{\pi}{2} \le t \le \frac{3\pi}{2}$



$$\int_\gamma z^{\frac{1}{2}}\, dz = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(e^{it}\right)^{\frac{1}{2}} ie^{it}\, dt = \left[\frac{2}{3} e^{i\frac{3}{2}t}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$
$$= \frac{2}{3}\left(e^{\frac{9\pi i}{4}} - e^{\frac{3\pi i}{4}}\right) = \frac{2}{3}\frac{1}{\sqrt{2}}\left[(1+i) - (-1+i)\right] = \frac{2\sqrt{2}}{3}$$

(c)

$z(t) = it, \quad 0 \le t \le 1, \quad \frac{dz}{dt} = i$



$$\int_\gamma \sinh z\, dz = \int_0^1 i \sinh it\, dt = \int_0^1 \frac{i}{2}(e^{it} - e^{-it})\, dt$$
$$= \int_0^1 -\sin t\, dt = [\cos t]_0^1 = \cos(1) - 1$$