Question
Using the matrices

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

show that the matrices $(C^T B)A^T$ and $C^T(BA^T)$ exist and are equal

Answer

$$\begin{array}{ccc} C^T & \text{is} & 2\times 3 \\ B & \text{is} & 3\times 3 \end{array} \Rightarrow \quad C^T B \text{ is } 3\times 3$$

A is $2\times 3 \Rightarrow A^T$ is 3×2
Similarly BA^T is 3×2 and C^T is $2\times 3 \Rightarrow C^T(BA^T)$ is 2×2
Hence both exist

$$C^T B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1+5 & 3+5 \\ 2 & 2+6 & 4+6 \end{pmatrix} = \begin{pmatrix} 1 & 6 & 8 \\ 2 & 8 & 10 \end{pmatrix}$$

$$\begin{aligned} (C^T B)A^T = \begin{pmatrix} 1 & 6 & 8 \\ 2 & 8 & 10 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} &= \begin{pmatrix} 1+12+24 & 4+30+48 \\ 2+16+30 & 8+40+60 \end{pmatrix} \\ &= \begin{pmatrix} 37 & 82 \\ 48 & 108 \end{pmatrix} \end{aligned}$$

$$BA^T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 1+2 & 4+5 \\ 3 & 6 \\ 2+3 & 5+6 \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 3 & 6 \\ 5 & 11 \end{pmatrix}$$

$$\begin{aligned} C^T(BA^T) = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 3 & 9 \\ 3 & 6 \\ 5 & 11 \end{pmatrix} &= \begin{pmatrix} 3+9+25 & 9+18+55 \\ 6+12+30 & 18+24+66 \end{pmatrix} \\ &= \begin{pmatrix} 37 & 82 \\ 48 & 108 \end{pmatrix} \end{aligned}$$

Therefore $(C^T B)A^T = C^T(BA^T)$ as required