Question

Find the vector equation of the line that passes through the point A with position vector $\mathbf{a} = -2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and is normal to both the vectors $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{c} = -\mathbf{i} + \mathbf{j} - \mathbf{k}$.

Answer

The direction vector of the line must be normal to
$\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{c} = -\mathbf{i} + \mathbf{j} - \mathbf{k}$
Let the direction vector be $\mathbf{d}$. Thus $\mathbf{d}$ must be parallel to either $\mathbf{b} \times \mathbf{c}$ or $\mathbf{c} \times \mathbf{b}$. Either one will do as we can just trivially alter the scalar parameter in the final vector equation of the line, if needs be.
So choose

$$\begin{aligned}
\mathbf{d} &= \mathbf{b} \times \mathbf{c} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -1 & 1 & -1 \end{vmatrix} \\
&= \mathbf{i}(2 \times -1) - \mathbf{i}(1 \times 3) - \mathbf{j}(1 \times -1) \\
&\quad +\mathbf{k}(1 \times 1) + \mathbf{j}(3 \times -1) - \mathbf{k}(2 \times -1) \\
&= -2\mathbf{i} - 3\mathbf{i} + \mathbf{j} + \mathbf{k} - 3\mathbf{j} + 2\mathbf{k} \\
&= -5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}
\end{aligned}$$

Thus if $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ where $\mathbf{a}$ is a point on the line, we have

$$\underline{\mathbf{r} = -2\mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(-5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})}$$