Question
Determine unit vectors that are normal to both vectors $\mathbf{a}$ and $\mathbf{b}$ when:

(i) $\mathbf{a} = 3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$ $\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

(ii) $\mathbf{a} = -4\mathbf{i} + 2\mathbf{k}$ $\mathbf{b} = \mathbf{j} - 3\mathbf{k}$

Are the results unique?

Answer
First a vector normal to both $\mathbf{a}$ and $\mathbf{b}$ is given by $\mathbf{a} \times \mathbf{b}$. The corresponding unit vector is $\dfrac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$

(i)

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 5 & -2 \\ 1 & 1 & 1 \end{vmatrix} \\
&= \mathbf{i}(5 \times 1) - \mathbf{i}(1 \times -2) - \mathbf{j}(3 \times 1) \\
&\quad +\mathbf{k}(3 \times 1) + \mathbf{j}(1 \times -2) - \mathbf{k}(1 \times 5) \\
&= 5\mathbf{i} + 2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k} - 2\mathbf{j} - 5\mathbf{k} \\
&= 7\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}
\end{aligned}$$

Now $|\mathbf{a} \times \mathbf{b}| = \sqrt{7^2 + (-5)^2 + (-2)^2} + \sqrt{49 + 25 + 4} = \sqrt{78}$

Thus $\underline{\dfrac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|} = \dfrac{7}{\sqrt{78}}\mathbf{i} - \dfrac{5}{\sqrt{78}}\mathbf{j} - \dfrac{2}{\sqrt{78}}\mathbf{k}}$

(ii)

$$\begin{aligned}
&\mathbf{a} \times \mathbf{b} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 0 & 2 \\ 0 & 1 & -3 \end{vmatrix} \\
&= \mathbf{i}(0 \times -3) - \mathbf{i}(1 \times 2) - \mathbf{j}(-3 \times -4) \\
&\quad +\mathbf{k}(-4 \times 1) + \mathbf{j}(0 \times 2) - \mathbf{k}(0 \times 0) \\
&= -2\mathbf{i} - 12\mathbf{j} - 4\mathbf{k}
\end{aligned}$$

Now $|\mathbf{a} \times \mathbf{b}| = \sqrt{2^2 + 12^2 + 4^2} + \sqrt{4 + 144 + 16} = \sqrt{164}$

Thus $\underline{\dfrac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|} = -\dfrac{2}{\sqrt{164}}\mathbf{i} - \dfrac{12}{\sqrt{164}}\mathbf{j} - \dfrac{4}{\sqrt{164}}\mathbf{k}}$

These results are not unique, you could have the negative sign of them. Also, there are two vectors perpendicular to any pair (these vectors being $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ [$= -\mathbf{a} \times \mathbf{b}$]).