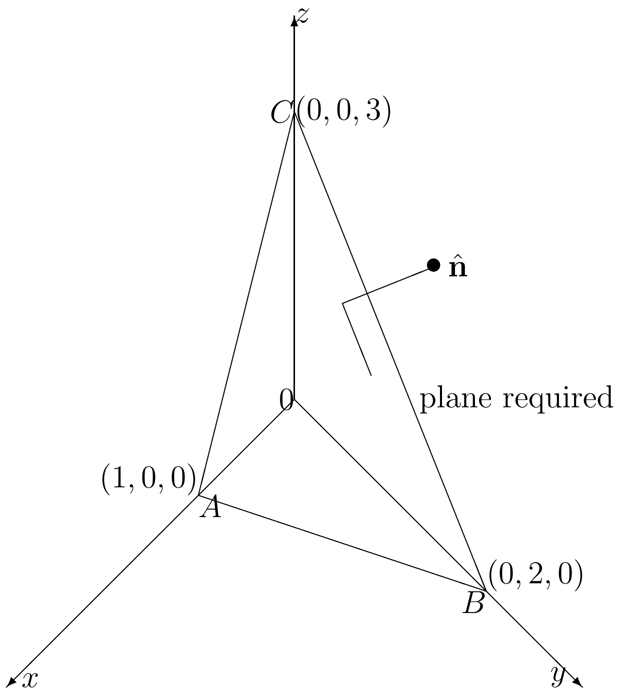
Question

Find the Cartesian and vector equations of the plane cutting the x axis at (1,0,0), the y axis at (0,2,0) and the z axis at (0,0,3).

Answer



Let

$$\begin{aligned}\mathbf{OA} &= (1,0,0) = \mathbf{i}\\ \mathbf{OB} &= (0,2,0) = 2\mathbf{j}\\ \mathbf{OC} &= (0,0,3) = 3\mathbf{k}\end{aligned}$$

So two vectors in the plane are, e.g.,
$\mathbf{AB}$ and $\mathbf{BC}$
$\mathbf{AB} = \mathbf{AO} + \mathbf{OB} = -\mathbf{i} + 2\mathbf{j}$
$\mathbf{BC} = \mathbf{BO} + \mathbf{OC} = -2\mathbf{j} + 3\mathbf{k}$
A vector to a point in the plane is $\mathbf{OA} = \mathbf{i}$.
Thus a vector equation is

$$\begin{aligned}\mathbf{r} &= \mathbf{OA} + \lambda(\mathbf{AB}) + \mu(\mathbf{BC})\\ &= \mathbf{i} + \lambda(-\mathbf{i} + 2\mathbf{j}) + \mu(-2\mathbf{j} + 3\mathbf{k})\end{aligned}$$

Cartesian equation is derived as in Q11.
Let unit vector normal to plane ABC be $\hat{\mathbf{n}}$.

$$\hat{\mathbf{n}} = \frac{\mathbf{AB} \times \mathbf{BC}}{|\mathbf{AB} \times \mathbf{BC}|}$$

$$\begin{aligned}
&\mathbf{AB} \times \mathbf{BC} \\
= &\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ 0 & -2 & 3 \end{vmatrix} \\
= &\mathbf{i}(2 \times 3) - \mathbf{i}(-2 \times 0) - \mathbf{j}(-1 \times 3) \\
&+\mathbf{k}(-1 \times -2) + \mathbf{j}(0 \times 0) - \mathbf{k}(0 \times 2) \\
= &6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}
\end{aligned}$$

$|\mathbf{AB} \times \mathbf{BC}| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{36 + 9 + 4} = \sqrt{49} = 7$
so $\hat{\mathbf{n}} = \frac{6}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{2}{7}\mathbf{k}$
We now need the perpendicular distance, d, from 0 to the plane. This is given by $d = \mathbf{r} \cdot \hat{\mathbf{n}}$ where $\mathbf{r}$ is the position vector if any point in the plane; say $\mathbf{OA} = \mathbf{i}$
Thus $d = \mathbf{i} \cdot \left(\frac{6}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{2}{7}\mathbf{k}\right) = \frac{6}{7}$

Thus we have that
$$\begin{aligned}
&(x, y, z) \cdot \hat{\mathbf{n}} = d \\
\text{so} \quad &(x, y, z) \cdot \left(\frac{6}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{2}{7}\mathbf{k}\right) = \frac{6}{7} \\
\Rightarrow \quad &\underline{6x + 3y + 2z = 6}
\end{aligned}$$