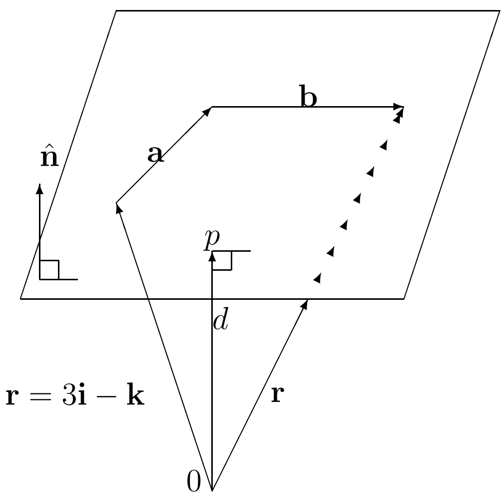
Question

Find the vector and Cartesian equation of the plane containing the point $3\mathbf{i} - \mathbf{k}$ and also containing the vectors $\mathbf{a},\ \mathbf{b}$ where
$\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$,
$\mathbf{b} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

Answer



$\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$
$\mathbf{b} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
Vector equation is
$\mathbf{r} = \mathbf{c} + \lambda\mathbf{a} + \mu\mathbf{b}$
$\Rightarrow \underline{\mathbf{r} = 3\mathbf{i} - \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) + \mu(-\mathbf{j} + 2\mathbf{j} + 2\mathbf{k})}$
Cartesian equation is given by the set of points $\mathbf{r} = (x, y, z)$ satisfying $(x, y, z) \cdot \hat{\mathbf{n}} = d$ where $\hat{\mathbf{n}}$ is a unit vector to the plane and d is the perpendicular distance from the origin(—OP—). First we need $\hat{\mathbf{n}}$. This is given by $\dfrac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$, since $\mathbf{a},\ \mathbf{b}$ lie in the plane.

$$\begin{aligned}
& \mathbf{a} \times \mathbf{b} \\
= & \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ -1 & 2 & 2 \end{vmatrix} \\
= & \mathbf{i}(2 \times 2) - \mathbf{i}(2 \times 1) - \mathbf{j}(1 \times 2) \\
& +\mathbf{k}(1 \times 2) + \mathbf{j}(-1 \times 1) - \mathbf{k}(-1 \times 2) \\
= & 4\mathbf{i} - 2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} - \mathbf{j} + 2\mathbf{k} \\
= & 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}
\end{aligned}$$

$\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|} = \frac{2\mathbf{i}-3\mathbf{j}+4\mathbf{k}}{\sqrt{2^2+3^2+4^2}} = \frac{2}{\sqrt{29}}\mathbf{i} - \frac{3}{\sqrt{29}}\mathbf{j} + \frac{4}{\sqrt{29}}\mathbf{k}$

What is d? It's given by $\mathbf{c}\cdot\hat{\mathbf{n}}$ (see diagram above) $\Rightarrow$

$$\begin{aligned} d &= (3\mathbf{i}-\mathbf{k})\cdot\left(\frac{2}{\sqrt{29}}\mathbf{i} - \frac{3}{\sqrt{29}}\mathbf{j} + \frac{4}{\sqrt{29}}\mathbf{k}\right) \\ &= \frac{6}{\sqrt{29}} - \frac{4}{\sqrt{29}} = \frac{2}{\sqrt{29}} \end{aligned}$$

Thus we have

$(x,y,z)\cdot\left(\frac{2}{\sqrt{29}}, -\frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}\right) = \frac{2}{\sqrt{29}}$

$\Rightarrow \frac{2x}{\sqrt{29}} - \frac{2y}{\sqrt{29}} + \frac{4z}{\sqrt{29}} = \frac{2}{\sqrt{29}}$

$\Rightarrow \underline{2x-3y+4z=2}$ $(\star)$

Check: from vector equation above taking $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

Components:

$$\begin{aligned} x &= (3+\lambda-\mu) \\ y &= (2\lambda+2\mu) \\ z &= (-1+\lambda+2\mu) \end{aligned}$$

Substitute into LHS of $(\star)$:

$$\begin{aligned} & 2(3+\lambda-\mu) - 3(2\lambda+2\mu) + 4(-1+\lambda+2\mu) \\ =\ & 6+2\lambda-2\mu-6\lambda-6\mu-4+4\lambda+8\mu \\ =\ & 2 \\ =\ & RHS\ (\star)\surd \end{aligned}$$