Question

Suppose that a r.v. X has the mgf

$$m(t) = e^{t^2+3t} \quad \text{for} \quad -\infty < t < \infty.$$

Find $E\{[x - E(X)]^r\}$, the rth central moment of X, for $r = 1, 2, \cdots$.
Does X have a normal distribution? Give your reasoning.

Answer

It is known that $X \sim N(\mu, \sigma^2)$ if and only if $M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$
Therefore $M_X(t) = e^{3t+t^2}$ is the mgf of $N(\mu = 3, \sigma^2 = 2)$
Let $Y = X - \mu$
Therefore $E\{[X - E(X)]^r\} = E(Y^r)$
But

$$\begin{aligned} M_Y(t) &= e^{-t\mu} e^{\mu t + t^2} \\ &= e^{t^2} \\ &= \sum_{k=0}^{\infty} \frac{t^{2k}}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(2k)!}{k!} \cdot \frac{t^{2k}}{(2k)!} \end{aligned}$$

Therefore $E(Y^r) = \begin{cases} 0 & \text{if } r \text{ is odd} \\ \frac{(2k)!}{k!} & \text{if } r = 2k \end{cases}$