Question

Suppose that X has the standard normal distribution, i.e.

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad -\infty < x < \infty.$$

Derive the moment generating function of $Y = X^2$. What distribution does Y follow?

Answer

$$\begin{aligned}
M_X(t) &= E(e^{tY}) \\
&= E\{e^{tX^2}\} \\
&= \int_{-\infty}^{\infty} e^{tx^2} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\, dx \\
&= \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}x^2(1-2t)}}{\sqrt{2\pi}}\, dx \\
&= \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\frac{x^2}{b^2}}}{\sqrt{2\pi}}\, dx = b \\
&\quad \text{where } b^2 = \frac{1}{1-2t} \quad (\text{if } 1-2t > 0 \Rightarrow t < \frac{1}{2}) \\
&= \left(\frac{1}{1-2t}\right)^{\frac{1}{2}} \text{ if } t < \frac{1}{2}.
\end{aligned}$$

Since the above is the mgf of the χ^2 distribution with 1 degree of freedom we can conclude that $Y = X^2$ follows the χ^2 distribution with 1 degree of freedom.