Question

Suppose that a random variable X has the pmf

$$f(x) = C\frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 1, 2, ..., \lambda > 0.$$

Note that X takes values 1,2,3, ... Find C and $E(X)$.

Answer

Since X is discrete and $f(x)$ is a pmf we must have

$$\begin{aligned}
& \sum_{x=1}^{\infty} f(x) = 1 \\
\Rightarrow \quad & \sum_{x=1}^{\infty} c\frac{e^{-\lambda}\lambda^x}{x!} = 1 \\
\Rightarrow \quad & ce^{-\lambda}\left\{\sum_{x=1}^{\infty}\frac{\lambda^x}{x!} + \frac{\lambda^0}{0!} - \frac{\lambda^0}{0!}\right\} = 1 \\
\Rightarrow \quad & ce^{-\lambda}\left\{e^{-\lambda} - 1\right\} = 1 \\
\Rightarrow \quad & c\left\{1 - e^{-\lambda}\right\} = 1 \\
\Rightarrow \quad & c = \frac{1}{1 - e^{-\lambda}}
\end{aligned}$$

$$\begin{aligned}
E(X) &= \frac{e^{-\lambda}}{1 - e^{-\lambda}}\sum_{x=1}^{\infty}\frac{x\lambda^x}{x!} \\
&= \frac{e^{-\lambda}}{1 - e^{-\lambda}}\sum_{x=1}^{\infty}\frac{\lambda^x}{(x-1)!} \\
&= \frac{e^{-\lambda}}{1 - e^{-\lambda}}\lambda\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!} \\
&= \frac{e^{-\lambda}}{1 - e^{-\lambda}}\lambda\sum_{t=0}^{\infty}\frac{\lambda^t}{t!} \\
&= \frac{e^{-\lambda}}{1 - e^{-\lambda}}\lambda e^{\lambda} \\
&= \frac{\lambda}{1 - e^{-\lambda}}.
\end{aligned}$$