Question

A man with n keys wants to open his door and tries the keys at random. Exactly one key will open the door. Let X denote the number of trials required to open the door for the first time. Find $E(X)$ if

(a) unsuccessful keys are not eliminated from further selections

(b) unsuccessful keys are eliminated

Answer

Let X be the number of trials needed to open the door. Let 'S' denote success i.e. the door is opened and 'F' denote failure for each trial.

(a) The event $X = x$ is equivalent to the event $\underbrace{F\ F\ F\ \ldots\ F}_{x-1 \text{ times}} S$

Also $P(S) = \dfrac{1}{n}$ and $P(F) = \dfrac{n-1}{n}$.

X has the geometric distribution with $p = P(S) = \dfrac{1}{n}$.

Therefore $E(X) = \dfrac{1}{p} = \dfrac{1}{\frac{1}{n}} = n$

(b) If unsuccessful keys are eliminated then it can take at most n attempts to open the door with the following probabilities:

$$
\begin{aligned}
P(X=1) &= \frac{1}{n} \\
P(X=2) &= \frac{n-1}{n} \cdot \frac{1}{n-1} = \frac{1}{n} \\
P(X=3) &= \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \frac{1}{n-2} = \frac{1}{n} \\
&\vdots \\
P(X=n) &= \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \frac{n-3}{n-2} \ldots \cdot \frac{1}{2} \cdot 1 = \frac{1}{n}.
\end{aligned}
$$

Therefore $E(X) = 1 \cdot \dfrac{1}{n} + 2 \cdot \dfrac{1}{n} + \ldots\ n \cdot \dfrac{1}{n} = \dfrac{n(n+1)}{2n} = \dfrac{n+1}{2}$.