**Question**

If $T : z \to \frac{z-i}{z+i}$ compute $T^{-1}$ and hence or otherwise show that $T$ maps the upper half plane $\{z|\mathrm{Im}(z) > 0\}$ onto the unit disc $D = \{z|\,|z| < 1\}$. Show further that $T$ maps the positive real axis to the lower half of the unit circle and the negative real axis to the upper half of the unit circle.

Show that $z \to e^{\pi z}$ maps the strip $\{0 < \mathrm{Im}(z) < 1\}$ conformally onto the upper half plane and find the images of the lines $\mathrm{Im}(z) = 0$ and $\mathrm{Im}(z) = 1$ under this transformation.

Hence find a conformal transformation which maps $D$ onto this strip and takes the lower half circle to the real axis and the upper half circle to the line $\mathrm{Im}(z) = 1$.

**Answer**

$w = \dfrac{z-i}{z+i} \qquad z = i\dfrac{1+w}{1-w}$

If $z$ is real then $w = \dfrac{x-i}{x+i} \qquad \bar{w} = \dfrac{x+i}{x-i}$, so $w\bar{w} = 1$

so $w$ is on the unit circle.

If $w = e^{i\theta}$, then $z = i\dfrac{(1+e^{i\theta})}{1-e^{i\theta}} = i\dfrac{e^{-i\frac{\theta}{2}} + e^{i\frac{\theta}{2}}}{e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}} = -\cot\dfrac{1}{2}\theta$ - real

so the real axis maps to $|z| = 1$ and vice versa.

$z = i \to w = 0$ so $U$ maps to $D$.

Now $-\pi < \theta < 0 \Leftrightarrow -\cot\frac{1}{2}\theta > 0$

And $0 < \theta < \pi \Leftrightarrow -\cot\frac{1}{2}\theta < 0$

So the lower half of the circle $\to$ positive real axis.

and the upper half of the circle $\to$ negative real axis.

$w = e^{\pi z}$ - analytic

$z = x \Rightarrow w = e^{\pi x}$ positive real axis

$z = x + i \Rightarrow w = e^{\pi x + i} = -e^{\pi x}$ negative real axis

$z = x + iy_0 \Rightarrow w = e^{\pi x}e^{i\pi y_0}$ which is a ray from 0 at angle $\pi y_0$, which goes from 0 to $\pi$ as $y_0$ goes from 0 to 1. So the strip maps conformally to the upper half plane.

$T^{-1}$ maps $D$ to $U$.

$z = \dfrac{1}{\pi}\log w$ maps $U$ to $S$

$z = \dfrac{1}{\pi}\log i\dfrac{1+w}{1-w}$ maps $D$ to $S$

and maps the halves of the circle in the required fashion.