Question

i) Let $h(z) = \dfrac{1}{(z-2)(z+3)}$.

Find the Laurent series expansion of $f(z)$ in powers of z, valid in the annulus $2 < |z| < 3$ and obtain the coefficient of z^n explicitly for $n = -2, -1, 0, 1, 2$.

ii) Let $f(z)$ and $g(z)$ be analytic functions in a neighbourhood of $z = a$. Let $f(z)$ have a zero of order k at $z = a$ and $g(z)$ have a zero of order l at $z = a$. If $k > l$ show that $F(z) = \frac{f(z)}{g(z)} \quad (z \neq a)$ has a removable singularity at $z = a$ and that if we extend its definition to $z = a$ by defining $F(a) = 0$ then F has a zero of order $k - l$ at $z = a$.

Describe the nature of $F(z)$ near $z = a$ if $k < l$.

If $k = l$ show that $F(z)$ again has a removable singularity at $z = a$. If $g'(a) \neq 0$ prove that $l = 1$ and

$$\lim_{z \to a} \frac{f(z)}{g(z)} = \frac{f'(a)}{g'(a)}.$$

Answer

i) $\dfrac{1}{(z-2)(z-3)} = \dfrac{1}{z-3} - \dfrac{1}{z-2} = \dfrac{-1}{3\left(1 - \frac{z}{3}\right)} - \dfrac{1}{z\left(1 - \frac{2}{z}\right)}$

For $2 < |z| < 3$, $\left|\dfrac{z}{3}\right| < 1$ and $\left|\dfrac{2}{z}\right| < 1$, so using the binonmial theorem gives

$$-\frac{1}{3}\left(1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \cdots\right) - \frac{1}{z}\left(1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \cdots\right)$$

$$= \cdots - \frac{2^n}{z^{n+1}} - \cdots - \frac{2}{z^2} - \frac{1}{z} - \frac{1}{3} - \frac{z}{3^2} - \frac{z^2}{3^3} - \cdots - \frac{z^n}{3^{n+1}} - \cdots$$

ii) $f(z) = (z-a)^k f^*(z)$ and $g(z) = (z-a)^l g^*(z)$ where f^* and g^* are analytic in a neighbourhood of a and non-zero at $z = a$.

$F(z) = \dfrac{f(z)}{g(z)} = (z-a)^{k-l}\dfrac{f^*(z)}{g^*(z)}$

If $k > l$, $F(z) \to 0$ as $z \to a$, a removable singularity.

If $k < l$, $F(z)$ has a pole of order $l - k$ at $z = a$.

If $k = l$, $F(z) = \dfrac{f^*(z)}{g^*(z)} \to \dfrac{f^*(a)}{g^*(a)} \neq 0$, so F has a removable singularity.

Now if $l = 1$, $g(z) = g'(a)(z - a) + \cdots$ (by Taylor's theorem)

So $g^*(a) = g'(a)$, and similarly $f^*(a) = f'(a)$.

So $\lim_{z \to a} \dfrac{f(z)}{g(z)} = \dfrac{f'(a)}{g'(a)}$