Question

i) State the Maximum Modulus Principle for a function $f(z)$ that is analytic within a simple closed contour γ and continuous on γ.

Show that if $f(z) \neq 0$ for all z within and on γ, then the minimum value of $|f(z)|$ cannot be achieved in the interior of γ.

Hence find the maximum and minimum values of $|e^{z^2}|$ in the disc $\{z|\,|z| \leq 1\}$, and find where these values are attained.

ii) Suppose that $f(z)$ and $g(z)$ are analytic in the disc $\{z|\,|z| < 1\}$ and continuous on $|z| = 1$. Suppose that f and g do not vanish for $|z| \leq 1$, and that $|f(z)| = |g(z)|$ whenever $|z| = 1$. Prove that if $f(0)$ and $g(0)$ are positive real numbers then $f(z) = g(z)$ whenever $|z| \leq 1$.

Answer

i) First part bookwork.

Now $e^{z^2} \neq 0$, so $|e^{z^2}|$ has max and min in D on the boundary.

Let $z = e^{i\theta}$, so $e^{z^2} = e^{2i\theta}$, and $|e^{z^2}| = c^{\cos 2\theta}$

so this is maximal where $\cos 2\theta = +1$ i.e. $\theta = 0, \pi$ i.e. max e.

and this is minimal where $\cos 2\theta = -1$ i.e. $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ i.e. min e^{-1}.

ii) Let $h(z) = \dfrac{f(z)}{g(z)}$ and $k(z) = \dfrac{g(z)}{f(z)}$.

Both h and k are analytic in $|z| < 1$ since f and g are, and are non-zero. The max of $|h|$ and $|k|$ are both achieved on the boundary, and in fact $|h| = |k| = 1$ on the boundary.

So $\dfrac{f(0)}{g(0)} \leq 1$ and $\dfrac{g(0)}{f(0)} \leq 1$. Thus $f(0) = g(0) = 1$.

Thus h has a local maximum at 0 and is therefore constant in D

i.e. $f(z) \equiv g(z)$ in D.