Question

Throughout this question the branch of $\log z$ whose imaginary part v satisfies $-\pi < v \leq \pi$ is used.

i) Let γ be the circle $|z| = R$, where $R > 1$. Stating carefully any inequalities concerning integrals you use prove that

$$\left| \int_\gamma \frac{\log z dz}{z^2} \right| \leq 2\pi \left(\frac{\pi + \log R}{R} \right)$$

ii) Evaluate $\displaystyle\int_\delta \log z dz$, where δ is the upper half of the unit circle from $z = 1$ to $z = -1$.

iii) Let n denote the circle with centre $\dfrac{1+\sqrt{3}i}{2}$ and radius $\frac{1}{2}$. Use the Cauchy Integral Formula to evaluate

$$\int_n \frac{\log z dz}{z - \left(\frac{1}{2} + \frac{\sqrt{3}i}{2} \right)}$$

Answer

i) On γ, $|\log z| = |\log R + i\theta| \leq \log R + \pi$

since $|z| = R$, $\left| \dfrac{\log z}{z^2} \right| \leq \dfrac{\log R + \pi}{R^2}$

$l(\gamma) = 2\pi R$, so $\left| \displaystyle\int_\gamma \frac{\log z}{z^2} dz \right| \leq 2\pi \left(\dfrac{\log R + \pi}{R} \right)$

ii) On δ, $z = e^{i\theta}$ and $\log z = i\theta$, $0 \leq \theta \leq \pi$

So $\displaystyle\int_\delta \log z dz = \int_0^\pi i\theta i e^{i\theta} d\theta = 2 - \pi i$ (by parts)

iii) n lies in the first quadrant so $\log z$ is analytic inside and on n

So $\displaystyle\int_n \frac{\log z}{z - \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i \right)} = \log \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i \right) = \log 1 + i\frac{\pi}{3} = i\frac{\pi}{3}$